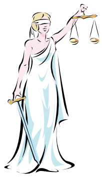


3.2 The scalar & vector products of vectors

Content

3.2.1 The scalar and vector products of vectors

Learning Outcomes

Include:

(a) Concepts of scalar product and vector product of vectors

(b) Calculation of the magnitude of a vector and the angle between two directions

(c) Calculation of the area of triangle or parallelogram

(d) Geometrical meanings of $|\mathbf{a} \bullet \mathbf{b}|$ and $|\mathbf{a} \times \mathbf{b}|$, where $\mathbf{b}$ is a unit vector

Exclude

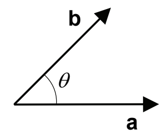
- triple products $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

3.2.1 The scalar & vector products of vectors

Define

Scalar product

The **scalar product** of two non-zero vectors **a** and **b**, written as $\mathbf{a}\cdot\mathbf{b}$ is given by $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between **a** and **b**.



- $\mathbf{a}\cdot\mathbf{b}$ is <u>positive</u> when $\cos\theta$ is <u>positive</u>, and $\mathbf{a}\cdot\mathbf{b}$ is <u>negative</u> when $\cos\theta$ is <u>negative</u>.

- If **a**, **b** and **c** are non-zero vectors, the laws of *scalar product* are:
  - *Commutative*: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
  - *Associative*: $(k\mathbf{a})\cdot\mathbf{b} = (k\mathbf{b})\cdot\mathbf{a} = k\mathbf{a}\cdot\mathbf{b}$
  - *Distributive*: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

- Other basic properties:

  ❶ If **a** and **b** are <u>parallel</u> vectors in the <u>same</u> direction $\Leftrightarrow$

  $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 0^\circ = |\mathbf{a}||\mathbf{b}|$$

  ❷ If **a** and **b** are <u>parallel</u> vectors in the <u>opposite</u> direction $\Leftrightarrow$

  $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 180^\circ = -|\mathbf{a}||\mathbf{b}|$$

  ❸ If **a** and **b** are <u>perpendicular</u> vectors $\Leftrightarrow$

  $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90^\circ = 0.$$

  ❹ If **a** and **b** are <u>equal</u> vectors, $\mathbf{a}=\mathbf{b}$ $\Leftrightarrow$

  $$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$$

- Mathematically, if $\mathbf{a} = x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$, then

$$\mathbf{a}\cdot\mathbf{b} = (x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})\cdot(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}) \text{ (vector form)}$$

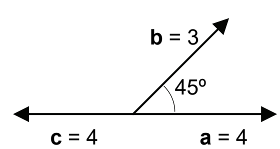
$$\text{or } \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\cdot\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \text{ (column vector form)}$$

$$= x_1x_2+y_1y_2+z_1z_2 \text{ (Cartesian form)}$$

Examples

① $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 40^\circ = 6\sqrt{2}$ (3 sf)

$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos 120^\circ = -6$



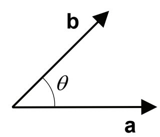
② The scalar product of the two vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\\6\end{pmatrix}$

$$= \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\6\end{pmatrix} = (1)(4)+(2)(5)+(3)(6) = 32 \quad \textbf{(ans)}$$

Angle between two vectors

- The angle θ, between two non-zero vectors **a** and **b**, can be calculated from its *scalar product*,

$$\mathbf{a}\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta .$$



- Thus the angle θ between the two vectors is given by

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}} \quad \text{or} \quad \boxed{\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)}$$

Examples [*Examined in* **2015**p2.2(i)]

① Find the angle between $\mathbf{a} = 2\mathbf{i}+4\mathbf{j}$ and $\mathbf{b} = -\mathbf{i}+2\mathbf{j}$, giving your answers to the nearest degree.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\begin{pmatrix}2\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\2\end{pmatrix}}{\sqrt{(2)^2+(4)^2}\sqrt{(-1)^2+(2)^2}} = \frac{-2+8}{\sqrt{20}\sqrt{5}} = 0.6$$

$\therefore \quad \theta = \cos^{-1}0.6 \;= 53.1^\circ \text{ (3sf)} = 53^\circ$ **(ans)**

② Find the angle between $\mathbf{a} = 2\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\mathbf{b} = -\mathbf{i}+2\mathbf{j}+6\mathbf{k}$, giving your answers to the nearest degree.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\begin{pmatrix}2\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\6\end{pmatrix}}{\sqrt{(2)^2+(4)^2+(-1)^2}\sqrt{(-1)^2+(2)^2+(-6)^2}}$$

$$= \frac{-2+8-6}{\sqrt{\ldots}\sqrt{\ldots}} = 0$$

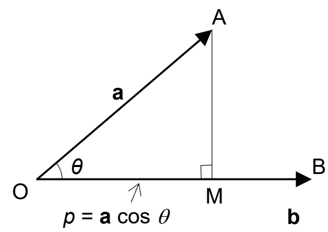
$\therefore \quad \theta = \cos^{-1}0 \;= 90^\circ$ (perpendicular to each other) **(ans)**

Geometrical interpretation of the scalar product

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{b}||\mathbf{a}|\cos\theta = |\mathbf{b}|\times OM$

$= |\mathbf{b}|\times p$

$= |\mathbf{b}| \times$ the length of projection of **a** on **b**



Thus, **length of projection** (p) of **a** on **b**,

$p = \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a}\cdot\hat{\mathbf{b}}|$ (where $\hat{\mathbf{b}}$ is a unit vector)

- In another words, the *projection* of vector **a** onto vector **b** is like a "shadow" of **a** being cast onto **b**, *i.e.*, $p = \mathbf{a}\cdot\hat{\mathbf{b}}$ where p is the length of the "shadow" cast and $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**.
- It may also be called the **component** of **a** in the *direction* of **b**.

Example **[***Examined in* **2012**p1.7, **2015**p2.2(ii)**]**

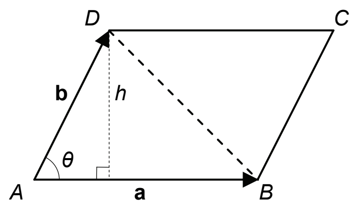
① Given two vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, find the length of projection of **a** on **b**.

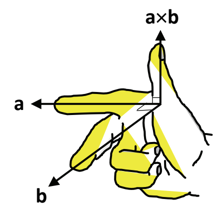
Length of projection of **a** on **b** = $\mathbf{a}\cdot\hat{\mathbf{b}} = \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}$

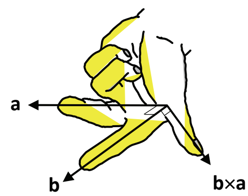
$$= \frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\-2\end{pmatrix}}{\sqrt{9+16+4}} = \frac{6}{\sqrt{29}}$$ **(ans)**

Vector product

> The **vector** (or **cross**) **product** of two non-zero vectors **a** and **b** (in that order), written as $\mathbf{a}\times\mathbf{b}$ is defined as the vector that has a direction, as given by the **right-hand rule**, that is <u>perpendicular</u> to both **a** and **b** and a magnitude given by $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$, where ϑ is the angle between **a** and **b**.







Given two vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ or $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ or $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

$$\mathbf{a}\times\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}\mathbf{i}+\begin{vmatrix}a_3&a_1\\b_3&b_1\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\mathbf{k}$$

$$=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

$$=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

- Do note that each component of the *vector product* does not contain the corresponding components of the operands (the vectors being operated upon, *i.e.*, **i**: $(a_2b_3 - a_3b_2)$ does not have a_1 nor b_1.

If **a**, **b** and **c** are non-zero vectors, the laws of *vector product* are:

- *Anti-Commutative*: $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
- *Distributive*: $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

  $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

Other basic properties:

❶ $\lambda(\mathbf{a}\times\mathbf{b}) = \lambda\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\lambda\mathbf{b}$

❷ If **a** and **b** are parallel vectors $\Leftrightarrow$

$\mathbf{a}\times\mathbf{b} = 0$

❸ If **a** and **b** are perpendicular vectors $\Leftrightarrow$

$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$

❹ If **a** and **b** are equal vectors, $\mathbf{a}=\mathbf{b}$ $\Leftrightarrow$

$\mathbf{a}\times\mathbf{a} = 0$

Example

① Find the vector product between the vectors $\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = (11)\mathbf{i}+(5)\mathbf{j}+(-7)\mathbf{k} \quad \textbf{(ans)}$$

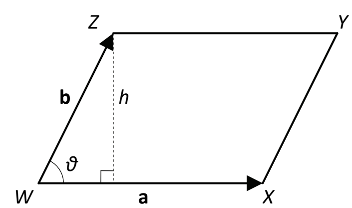
Some general results:

$\mathbf{i}\times\mathbf{j}=\mathbf{k}$	$\mathbf{j}\times\mathbf{k}=\mathbf{i}$	$\mathbf{k}\times\mathbf{i}=\mathbf{j}$
$\mathbf{j}\times\mathbf{i}=-\mathbf{k}$	$\mathbf{k}\times\mathbf{j}=-\mathbf{i}$	$\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
$\mathbf{i}\times\mathbf{i}=0$	$\mathbf{j}\times\mathbf{j}=0$	$\mathbf{k}\times\mathbf{k}=0$

Area of parallelogram

If vectors **a** and **b** represent two adjacent sides *WX* and *WZ* of a parallelogram *WXYZ*, then

Area of parallelogram $= |\mathbf{a}\times\mathbf{b}|$



Proof:

Area of parallelogram = base × height = $(WX)\cdot h$

$$\equiv |\mathbf{a}||\mathbf{b}|\sin\theta \equiv |\mathbf{a}\times\mathbf{b}| \equiv \left|\overrightarrow{WX}\times\overrightarrow{WZ}\right|$$

Example

① A parallelogram has two adjacent sides $\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.

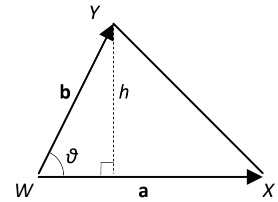
$$\text{Area} = |\mathbf{a}\times\mathbf{b}| = |(\mathbf{i}+2\mathbf{j}+3\mathbf{k})(2\mathbf{i}-3\mathbf{j}+\mathbf{k})|$$

$$= |\mathbf{i}(11)+\mathbf{j}(5)+\mathbf{k}(-7)| = \sqrt{195}\ \text{unit}^2 \quad \textbf{(ans)}$$

Area of triangle

If vectors **a** and **b** represent two sides *WX* and *WY* of a triangle *WXY*, then

$$\text{Area of triangle} = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$$



Example

① A triangle has two sides $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

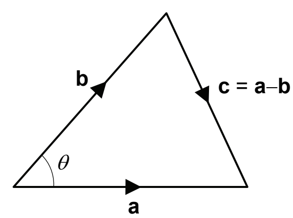
$$\text{Area} = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| = \tfrac{1}{2}\left|(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-3\mathbf{j}+\mathbf{k})\right|$$

$$= \tfrac{1}{2}\left|\mathbf{i}(11)+\mathbf{j}(5)+\mathbf{k}(-7)\right| = \sqrt{195}/2 \text{ unit}^2 \quad \textbf{(ans)}$$

Worked Examples

Example 1

Consider a triangle as shown and the scalar product $\mathbf{c}\cdot\mathbf{c} = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$, deduce the *cosine rule*.



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Solution

$$\mathbf{c}\cdot\mathbf{c} = |\mathbf{c}|^2 = c^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} \text{ (distributive)}$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} = a^2 + b^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta = a^2 + b^2 - 2ab\cos\theta$$

$$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos\theta \quad \textbf{(ans)}$$

Example 2

Given that *A*(2, 3, 4), *B*(–2, 1, 0) and *C*(4, 0, 2), find the exact value of $\sin\angle BAC$ and hence the area of the triangle *ABC*.

✯ ✯ ✯

Solution

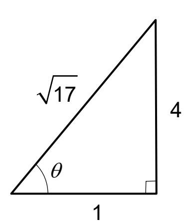
Position vectors *A*, *B* and *C*

a = 2**i** + 3**j** + 4**k**, **b** = –2**i** + **j**, **c** = 4**i** + 2**k**

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

$$\cos B\hat{A}C = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{-8+6+8}{\sqrt{4^2+2^2+4^2}\sqrt{2^2+3^2+2^2}} = \frac{1}{\sqrt{17}}$$



$$\sin B\hat{A}C = \sqrt{1-\cos^2 B\hat{A}C} = \sqrt{1-\frac{1}{17}} = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin B\hat{A}C = \frac{1}{2}\sqrt{36}\sqrt{17}\frac{4}{\sqrt{17}} = 12 \text{ units}^2 \quad \textbf{(ans)}$$

Example 3

Find a unit vector perpendicular to both **a** = 2**i** – **j** + 2**k** and **b** = 3**i** – 4**j** – 2**k**.

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Solution

Vector $\mathbf{a}\times\mathbf{b}$ is $\perp$ to both **a** and **b**

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}3\\-4\\-2\end{pmatrix} = \begin{pmatrix}(-1)(-2)-(-4)(2)\\2(3)-2(-2)\\(2)(-4)-(3)(-1)\end{pmatrix} = \begin{pmatrix}10\\10\\-5\end{pmatrix} = 5\begin{pmatrix}2\\2\\-1\end{pmatrix}$$

A unit vector perpendicular to both **a** and **b**

$$= \frac{2\mathrm{i}+2\mathrm{j}-\mathrm{k}}{\sqrt{2^2+2^2+1^2}} = \tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k}) \quad \textbf{(ans)}$$

Example 4 **[*Examined in* 2013p1.6(iii)]**

Given: $AN : NC = 3 : 4$. $\Delta OMC = \Delta ONC$. $\mathbf{c} = \lambda\mathbf{a}+\mu\mathbf{b}$.

Find λ in terms of μ. **[5]**

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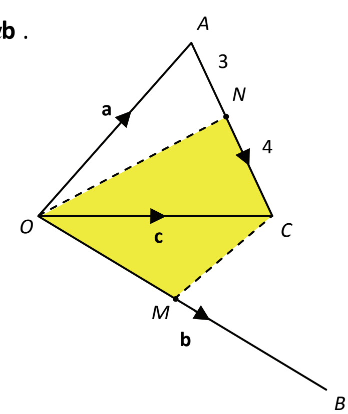

Solution

By ratio theorem, $\overrightarrow{ON} = \frac{4}{7}\mathbf{a}+\frac{3}{7}\mathbf{c}$

$$\text{Area of } \Delta ONC = \tfrac{1}{2}\left|\overrightarrow{ON}\times\overrightarrow{OC}\right| = \tfrac{1}{2}\left|\left(\tfrac{4}{7}\mathbf{a}+\tfrac{3}{7}\mathbf{c}\right)\times\mathbf{c}\right|$$

$$= \tfrac{1}{2}\left|\left(\tfrac{4}{7}\mathbf{a}+\tfrac{3}{7}(\lambda\mathbf{a}+\mu\mathbf{b})\right)\times(\lambda\mathbf{a}+\mu\mathbf{b})\right|$$

$$= \tfrac{1}{2}\left|\left(\left(\tfrac{4}{7}+\tfrac{3}{7}\lambda\right)\mathbf{a}+\tfrac{3}{7}\mu\mathbf{b}\right)\times(\lambda\mathbf{a}+\mu\mathbf{b})\right|$$

$$= \tfrac{1}{2}\left|\left[\left(\tfrac{4}{7}+\tfrac{3}{7}\lambda\right)\mathbf{a}\times\mu\mathbf{b}\right]+\left[\tfrac{3}{7}\mu\mathbf{b}\times\lambda\mathbf{a}\right]\right| = \tfrac{1}{2}\left|\left[\mu\left(\tfrac{4}{7}+\tfrac{3}{7}\lambda\right)\mathbf{a}\times\mathbf{b}\right]-\mu\left[\tfrac{3}{7}\lambda\mathbf{a}\times\mathbf{b}\right]\right|$$

$$= \tfrac{1}{2}\left|\left[\tfrac{4}{7}\mu(\mathbf{a}\times\mathbf{b})\right]\right| \quad ❶$$

$$\text{Area of } \Delta OMC = \tfrac{1}{2}\left|\overrightarrow{OM}\times\overrightarrow{OC}\right| = \tfrac{1}{2}\left|\tfrac{1}{2}\mathbf{b}\times\mathbf{c}\right| = \tfrac{1}{2}\left|\left(\tfrac{1}{2}\mathbf{b}\right)\times(\lambda\mathbf{a}+\mu\mathbf{b})\right|$$

$$= \tfrac{1}{2}\left|\left[\tfrac{1}{2}\lambda(\mathbf{b}\times\mathbf{a})\right]\right| = \tfrac{1}{2}\left|\left[\tfrac{1}{2}\lambda(\mathbf{a}\times\mathbf{b})\right]\right| \quad ❷$$

Comparing coefficients of ❶ and ❷,

$$\tfrac{4}{7}\mu = \tfrac{1}{2}\lambda \;\Rightarrow\; \therefore\; \lambda = \tfrac{8}{7}\mu \quad \textbf{(ans)} \quad \textbf{[5]}$$

Worked Problems

Example 1

Referred to the origin O, the position vectors of the points A and B are

$\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+4\mathbf{j}+\mathbf{k}$ respectively.

(a) Show that OA is perpendicular to OB. **[2]**

(b) Find the position vector of the point M $\left(\mathbf{m}=\overrightarrow{OM}\right)$ on the line segment AB such that $AM:MB=1:2$. **[3]**

(c) The point C has position vector $-4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$. Use a vector product to find the exact area of triangle OAC. **[4]**

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Solution

(a) Taking the dot product of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:

$$\overrightarrow{OA}\bullet\overrightarrow{OB}=(\mathbf{i}-\mathbf{j}+2\mathbf{k})\bullet(2\mathbf{i}+4\mathbf{j}+\mathbf{k})=(1)(2)+(-1)(4)+(2)(1)=0$$

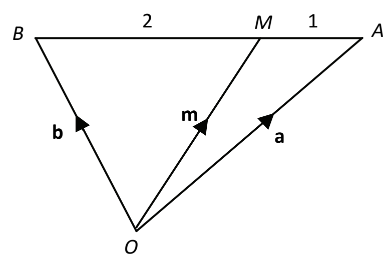
$\Rightarrow$ Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular. **(QED)**

(b) Using ratio theorem,

$$\mathbf{m}=\frac{2\mathbf{a}+\mathbf{b}}{2+1}=\tfrac{1}{3}(2\mathbf{a}+\mathbf{b})$$

$$=\tfrac{1}{3}\left(2(\mathbf{i}-\mathbf{j}+2\mathbf{k})+(2\mathbf{i}+4\mathbf{j}+\mathbf{k})\right)$$

$$=\tfrac{1}{3}(4\mathbf{i}+2\mathbf{j}+5\mathbf{k}) \quad \text{(ans)}$$



(c) Area of triangle OAC = $\tfrac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OC}\right|$

$$=\tfrac{1}{2}\left|(\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(-4\mathbf{i}+2\mathbf{j}+2\mathbf{k})\right|=\tfrac{1}{2}\left\|\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-4\\2\\2\end{pmatrix}\right\|$$

$$=\tfrac{1}{2}\left|\mathbf{i}(-2-4)+\mathbf{j}(-8-2)+\mathbf{k}(2-4)\right|=\tfrac{1}{2}\left|-6\mathbf{i}-10\mathbf{j}-2\mathbf{k}\right|$$

$$=\tfrac{1}{2}\sqrt{36+100+4}=\sqrt{35}\text{ units}^2 \quad \textbf{(ans)}$$

Example 2

The figure shows a right triangular prism with its bottom base lying in the xy – plane. The point P divides DE in the ratio 2:1. Give the co-ordinates of A, B, C, D, E and find the angle between the lines OP and AB.

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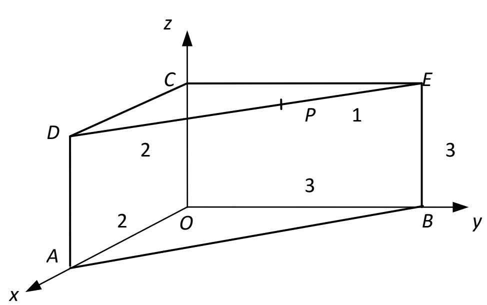

Solution

$A = (2, 0, 0)$ **(ans)** $B = (0, 3, 0)$ **(ans)** $C = (0, 0, 3)$ **(ans)**
$D = (2, 0, 3)$ **(ans)** $E = (0, 3, 3)$ **(ans)**

Given that $DP : PE = 2 : 1$,

$$\vec{OP} = \tfrac{1}{3}(2\vec{OE} + \vec{OD}) = \tfrac{1}{3}\left[\begin{pmatrix}0\\6\\6\end{pmatrix} + \begin{pmatrix}2\\0\\3\end{pmatrix}\right] = \tfrac{1}{3}\begin{pmatrix}2\\6\\9\end{pmatrix}$$

$\therefore\ P = \left(\tfrac{2}{3}, 2, 3\right)$

Let θ be the angle between OP and AB.

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}-2\\3\\0\end{pmatrix}$$

$$\cos\theta = \frac{\vec{OP}.\vec{AB}}{\left|\vec{OP}\right|\left|\vec{AB}\right|} = \frac{\begin{pmatrix}\frac{2}{3}\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\0\end{pmatrix}}{\sqrt{\left(\frac{2}{3}\right)^2 + (2)^2 + (3)^2}\sqrt{(-2)^2 + (3)^2 + 0^2}} = \frac{14}{11\sqrt{13}}$$

$\theta = 69.3°$ **(ans)**