3 Vectors

Content

3.1 Vectors in two and three dimensions

Include:

- Addition and subtraction of vectors, multiplication of a vector by a scalar, and their geometrical interpretations
- Use of notations such as $\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$
- Position vectors and displacement vectors
- Magnitude of a vector
- Unit vectors
- Distance between two points
- Angle between a vector and the x-, y- or z- axis
- Use of ration theorem in geometrical applications

3.2 The scalar and vector products of vectors

Include:

- Concepts of scalar product and vector product of vectors
- Calculation of the magnitude of a vector and the angle between two directions
- Calculation of the area of triangle or parallelogram
- Geometrical meanings of $|\mathbf{a}.\mathbf{b}|$ and $|\mathbf{a} \times \mathbf{b}|$, where $\mathbf{b}$ is a unit vector

Exclude triple products $\mathbf{a}.\mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

3.3 Three-dimensional geometry

Include:

- Vector and Cartesian equations of lines and planes
- Finding the distance from a point to a line or to a plane
- Finding the angle between two lines, between a line and a plane, or between two planes
- Relationships between
  - (i) two lines (coplanar or skew)
  - (ii) a line and a plane
  - (iii) two planes
  - (iv) three planes
- Finding the intersections of lines and planes

Exclude:

- Finding the shortest distance between two skew lines
- Finding an equation for the common perpendicular to two skew lines

3.1 Vectors in two & three dimensions

Solutions

1.

(a) $\overrightarrow{AB}=\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\overrightarrow{AC}=\begin{pmatrix}0\\1\\4\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\3\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=0$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\3\end{pmatrix}=0$ **(ans)**

(b) Mid-point of AB $=\begin{pmatrix}\frac{3}{2}\\1\\\frac{1}{2}\end{pmatrix}$

Vector equation of line:

$\mathbf{r}=\begin{pmatrix}\frac{3}{2}\\1\\\frac{1}{2}\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda\in\Re$ **(ans)**

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2. $\overrightarrow{AB}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

Line $l:\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix},\lambda\in\Re$

$\overrightarrow{CP}\bullet\begin{pmatrix}-1\\1\\0\end{pmatrix}=0$

$\overrightarrow{CP}=\begin{pmatrix}2-\lambda\\\lambda\\1\end{pmatrix}-\begin{pmatrix}3\\2\\0\end{pmatrix}=\begin{pmatrix}-1-\lambda\\-2+\lambda\\1\end{pmatrix}$

$\begin{pmatrix}-1-\lambda\\-2+\lambda\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\0\end{pmatrix}=0$

$1+\lambda-2+\lambda=0$

$\lambda=\frac{1}{2}$

$\overrightarrow{OP}=\begin{pmatrix}\frac{3}{2}\\\frac{1}{2}\\1\end{pmatrix}$

If ABCD is a rhombus, D is the point of reflection of point C in the line AB.

$\overrightarrow{CP}=\overrightarrow{PD}$

$\begin{pmatrix}-1-\frac{1}{2}\\-2+\frac{1}{2}\\1\end{pmatrix}=\overrightarrow{OD}-\overrightarrow{OP}$

$\overrightarrow{OD}=\begin{pmatrix}-\frac{3}{2}\\-\frac{3}{2}\\1\end{pmatrix}+\begin{pmatrix}\frac{3}{2}\\\frac{1}{2}\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **(ans)**

3.

(a) $\overrightarrow{OA}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$

$\overrightarrow{OB}=\begin{pmatrix}5\\-3\\0\end{pmatrix}$

$\overrightarrow{AB}=\begin{pmatrix}4\\-5\\3\end{pmatrix}$

$l_1:\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\-5\\3\end{pmatrix},\ \lambda\in\Re$ **(ans)**

(b) $l_2:\mathbf{r}=\begin{pmatrix}4\\-4\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\2\end{pmatrix},\ \mu\in\Re$

Substituting $\overrightarrow{OA}$ into l_2,

$\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}4+\mu\\-4-2\mu\\3+2\mu\end{pmatrix}$

$1=4+\mu$ – Equation 1

$2=-4-2\mu$ – Equation 2

$-3=3+2\mu$ – Equation 3

From Equation 1,

$\mu=-3$

Since $\mu=-3$ consistently, A lies on l_2. **(ans)**

(c) Let θ be the acute angle between l_1 and l_2.

$$\theta = \cos^{-1}\left|\frac{\begin{pmatrix}4\\-5\\3\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\2\end{pmatrix}}{\sqrt{(50)(9)}}\right| = \cos^{-1}\frac{20}{15\sqrt{2}}$$

$$= \cos^{-1}0.943 = 19.5^\circ \text{ (to the neaest } 0.1^\circ\text{)}$$

(ans)

(d) Length of projection of AC on l_1

$=$ AN (where N is the foot of the perpendicular from C to l_1)

$$= \frac{\left|\overrightarrow{AC}\bullet\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|}{\left|\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|} = \frac{\left|\begin{pmatrix}1\\-2\\2\end{pmatrix}\bullet\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|}{\left|\begin{pmatrix}4\\-5\\3\end{pmatrix}\right|}$$

$$= \frac{20}{\sqrt{50}} = 2\sqrt{2}$$

$$\left|\overrightarrow{AC}\right| = 3$$

Shortest distance

$$= CN = \sqrt{3^2 - \left(2\sqrt{2}\right)^2} = 1 \quad \textbf{(ans)}$$

4.

(a) When $\lambda = 1$,

$$\mathbf{r} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + \begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\4\end{pmatrix}$$

l_1 passes through point A. **(ans)**

(b) Let the acute angle be θ.

$$\cos\theta = \left|\frac{\begin{pmatrix}0\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}}{\sqrt{5}\sqrt{6}}\right| = \frac{3}{\sqrt{30}}$$

$\theta = 56.8^\circ$ **(ans)**

(c) Let M be any point on l_2.

$$\overrightarrow{OM} = \begin{pmatrix}1-\mu\\2\mu\\4-\mu\end{pmatrix}$$

$$\overrightarrow{EM} = \begin{pmatrix}1-\mu\\2\mu\\4-\mu\end{pmatrix} - \begin{pmatrix}1\\-4\\2\end{pmatrix} = \begin{pmatrix}-\mu\\4+2\mu\\2-\mu\end{pmatrix}$$

$$= \sqrt{6\mu^2 + 12\mu + 20}$$

$$= \sqrt{6(\mu+1)^2 + 14}$$

Shortest distance $= \sqrt{14}$ when $\mu = -1$. **(ans)**

5.

(a) Using Ratio Theorem,

$$\overrightarrow{OC} = \frac{2\overrightarrow{OB} + 3\overrightarrow{OA}}{2+3}$$

$$= \tfrac{2}{5}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \tfrac{3}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \tfrac{1}{5}\begin{pmatrix}9\\4\\2\end{pmatrix} \quad \textbf{(ans)}$$

(b) Direction vector, $\overrightarrow{AB} = \begin{pmatrix}3\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}2\\-3\\1\end{pmatrix}$

Vector equation of l:

$$\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\1\end{pmatrix} \quad \textbf{(ans)}$$

(c) $$\overrightarrow{OF} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}1+2\lambda\\2-3\lambda\\\lambda\end{pmatrix}$$

$$\overrightarrow{PF} = \begin{pmatrix}1+2\lambda\\2-3\lambda\\\lambda\end{pmatrix} - \begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}2\lambda\\3-3\lambda\\-5+\lambda\end{pmatrix}$$

Since $\overrightarrow{PF}$ is perpendicular to the line l,

$$\overrightarrow{PF}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}2\lambda\\3-3\lambda\\-5+\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix} = 0$$

$$4\lambda - 9 + 9\lambda - 5 + \lambda = 0$$

$$\lambda = 1$$

$$\overrightarrow{OF}=\begin{pmatrix}1+2\\2-3\\1\end{pmatrix}=\begin{pmatrix}3\\-1\\1\end{pmatrix}$$ **(ans)**

(d) $$\overrightarrow{PF}=\begin{pmatrix}2\\3-3\\-5+1\end{pmatrix}=\begin{pmatrix}2\\0\\-4\end{pmatrix}$$

Perpendicular distance from the point P to the line l

$$=\left|\overrightarrow{PF}\right|=\sqrt{2^2+(-4)^2}=\sqrt{20}=2\sqrt{5}\text{ units}$$

(ans)

(e) $\overrightarrow{OQ}=\overrightarrow{OP}+2\overrightarrow{PF}$

$$=\begin{pmatrix}1\\-1\\5\end{pmatrix}+2\begin{pmatrix}2\\3-3\\-5+1\end{pmatrix}=\begin{pmatrix}5\\-1\\-3\end{pmatrix}$$ **(ans)**

6.

(a) By Ratio Theorem,

$$\overrightarrow{OM}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Therefore,
Vector equation of line l :

$$\mathbf{r}=\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\ \lambda\in\Re$$ **(ans)**

(b) Since N lies on l ,

$$\overrightarrow{ON}=\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}$$ for a particular value of x

$$\overrightarrow{ON}=\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}\alpha\\\alpha-1\\-2\end{pmatrix}$$

Since AN is perpendicular to l ,

$$\overrightarrow{AN}\bullet\begin{pmatrix}1\\1\\0\end{pmatrix}=0$$

$\alpha+\alpha-1=0$

$\alpha=\frac{1}{2}$

Hence, $$\overrightarrow{ON}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$ **(ans)**

(c) Since AOCM is a parallelogram,

$\overrightarrow{OC}=\overrightarrow{AM}$

$$=\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}1\\0\\-2\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\-4\end{pmatrix}$$

$$\overrightarrow{AN}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}1\\-1\\-4\end{pmatrix}=\tfrac{1}{2}\overrightarrow{AC}$$

Since AN is parallel to AC and A is a common point, A, N and C are collinear. **(ans)**

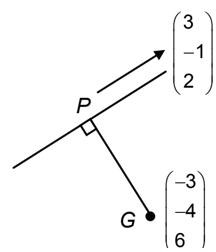
(d) Since AC is perpendicular to OM, AOCM is a rhombus.

Hence, $\dfrac{CO}{CM}=1$ **(ans)**

7.

(a) Vector equation of aircraft flight plan:

$$\text{r}=\begin{pmatrix}2\\-3\\4\end{pmatrix}+\mu\begin{pmatrix}3\\-1\\2\end{pmatrix}$$



$\overrightarrow{OG}=$ position vector of soldier

$$=\begin{pmatrix}-3\\-4\\6\end{pmatrix}$$

$$\overrightarrow{OA}=\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

$\overrightarrow{OP}=$ position vector of aircraft nearest to soldier

$$\overrightarrow{AP}=\frac{\left[\overrightarrow{AG}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}\right]\begin{pmatrix}3\\-1\\2\end{pmatrix}}{\left(\sqrt{9+1+4}\right)^2}=\frac{\left[\begin{pmatrix}-5\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}\right]\begin{pmatrix}3\\-1\\2\end{pmatrix}}{14}$$

$$=-\tfrac{5}{7}\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\begin{pmatrix}2\\-3\\4\end{pmatrix}-\tfrac{5}{7}\begin{pmatrix}3\\-1\\2\end{pmatrix}=\tfrac{1}{7}\begin{pmatrix}-1\\-16\\18\end{pmatrix}$$

Shortest distance $=\left|\overrightarrow{PG}\right|$

$$=\left|\begin{pmatrix}-3\\-4\\6\end{pmatrix}-\tfrac{1}{7}\begin{pmatrix}-1\\-16\\18\end{pmatrix}\right|=\tfrac{1}{7}\left|\begin{pmatrix}-20\\-12\\24\end{pmatrix}\right|=\tfrac{4}{7}\left|\begin{pmatrix}-5\\-3\\6\end{pmatrix}\right|$$

$$=\tfrac{4}{7}\sqrt{25+9+36}=4.78$$

Alternatively,

$$\overrightarrow{OP}=\begin{pmatrix}2+3p\\-3-p\\4+2p\end{pmatrix}$$

$$\overrightarrow{GP}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}=0$$

$$\begin{pmatrix}5+3p\\1-p\\-2+2p\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}=0$$

$15+9p-1+p-4+4p=0$

$p=-\tfrac{5}{7}$

$$\overrightarrow{OP}=\begin{pmatrix}2+3\left(-\tfrac{5}{7}\right)\\-3-\left(-\tfrac{5}{7}\right)\\4+2\left(-\tfrac{5}{7}\right)\end{pmatrix}=\tfrac{1}{7}\begin{pmatrix}-1\\-16\\18\end{pmatrix}$$

Shortest distance

$$=\sqrt{\left[5+3\left(-\tfrac{5}{7}\right)\right]^2+\left[1-\left(-\tfrac{5}{7}\right)\right]^2+\left[-2+2\left(-\tfrac{5}{7}\right)\right]^2}$$

$=4.78$ **(ans)**

(b) Equating,

$$\begin{pmatrix}-1-2\lambda\\1-5\lambda\\3+3\lambda\end{pmatrix}=\begin{pmatrix}2+3\mu\\-3-\mu\\4+2\mu\end{pmatrix}$$

$2\lambda+3\mu=-3$ – Equation 1

$5\lambda-\mu=4$ – Equation 2

$3\lambda-2\mu=1$ – Equation 3

Solving from Equations 2 and 3,

$\lambda=1,\ \mu=1$

Substituting into Equation1, Equation 3 is not satisfied.

Hence, the beam will not shine on the aircraft. **(ans)**

(c) $\overrightarrow{GP}=-\tfrac{4}{7}\begin{pmatrix}-5\\-3\\6\end{pmatrix}$

$\theta=$ angle between the original searchlight direction and $\overrightarrow{GP}$

$$\cos\theta=\frac{\begin{pmatrix}-2\\-5\\3\end{pmatrix}\bullet\begin{pmatrix}5\\3\\-6\end{pmatrix}}{\sqrt{4+25+9}\sqrt{25+9+36}}=0.834$$

$\theta=146.5°$ or $33.5°$ **(ans)**

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8. A unit vector in the direction of $\overrightarrow{AB}$

$$=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\frac{\begin{pmatrix}9\\7\\2\end{pmatrix}-\begin{pmatrix}-1\\2\\2\end{pmatrix}}{\sqrt{10^2+5^2}}=\frac{1}{5\sqrt5}\begin{pmatrix}10\\5\\0\end{pmatrix}=\frac{1}{\sqrt5}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

Let N be the foot of the perpendicular from O to line l.

Length of projection of OA on l

$$=AN=\frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|}=\frac{|-2-2+2|}{\sqrt{2^2+(-1)^2+1^2}}$$

$$=\frac{2}{\sqrt6}\text{ or }\frac{\sqrt6}{3}$$

Length of perpendicular from O to l

$$=\sqrt{OA^2-AN^2}$$

$$=\sqrt{\left(\sqrt{(-1)^2+2^2+2^2}\right)^2-\left(\frac{\sqrt6}{3}\right)^2}$$

$$=\frac{5}{\sqrt3}$$ **(ans)**

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9. $D\hat{C}E = 90^\circ$

$\overrightarrow{CD} \bullet \overrightarrow{CE} = 0$

$(\mathbf{d} - \mathbf{c}) \bullet (k\mathbf{d} - \mathbf{c}) = 0$

$k|\mathbf{d}|^2 - (k+1)(\mathbf{c} \bullet \mathbf{d}) + |\mathbf{c}|^2 = 0$

Given that $|\mathbf{c}| = 3$, $|\mathbf{d}| = 1$ and $C\hat{O}D = 60^\circ$,

$k(1) - (k+1)(3)(1)\cos 60^\circ + 9 = 0$

$k = 15$ **(ans)**

10.

(a) $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$

$\mathbf{b} = \begin{pmatrix} 6 \\ -5 \\ -1 \end{pmatrix}$

$\mathbf{c} = \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ -5 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Line AB is parallel to the vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ (shown) **(ans)**

(b) Vector equation of line AB, $\mathbf{r}$

$= \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\lambda \in \Re$ **(ans)**

(c) Let P be the foot of perpendicular from C to line AB, $\mathbf{r}$

$= \begin{pmatrix} 2\lambda \\ 1 - 2\lambda \\ -4 + \lambda \end{pmatrix}$ for some λ

$\overrightarrow{PC} = \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} - \begin{pmatrix} 2\lambda \\ 1 - 2\lambda \\ -4 + \lambda \end{pmatrix} = \begin{pmatrix} 4 - 2\lambda \\ 6 + 2\lambda \\ -5 - \lambda \end{pmatrix}$

$\overrightarrow{PC} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 0$

$\begin{pmatrix} 4 - 2\lambda \\ 6 + 2\lambda \\ -5 - \lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 0$

$\lambda = -1$

$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$ **(ans)**

(d) Length of projection

$= \left| \dfrac{\overrightarrow{OC} \bullet \overrightarrow{AB}}{|\overrightarrow{AB}|} \right|$

$= \left| \dfrac{\begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right|} \right| = \left| \dfrac{-15}{\sqrt{2^2 + (-2)^2 + 1}} \right| = 5$

(ans)

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11. $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, $\lambda \in \Re$

$L_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\mu \in \Re$

Since $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, L_1 is not parallel to L_2.

Consider $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$1 - 2\lambda = -2 + \mu$ – Equation 1

$2 + \lambda = 2 - 2\mu$ – Equation 2

$5 - \lambda = 3 + \mu$ – Equation 3

Solving Equations 2 and 3,

$\mu = -2$

$\lambda = 4$

Substituting into Equation 1,

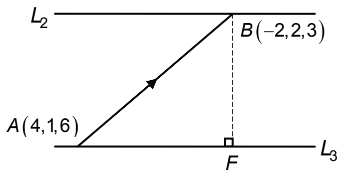
LHS $= -7$

RHS $= -4 \neq$ LHS

Hence, no values of λ and μ satisfy Equations 1, 2 and 3.

L_1 and L_2 do not intersect.

L_1 and L_2 are skew lines. (shown)



Note that $L_2 \parallel L_3$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-6)^2 + 1^2 + (-3)^2} = \sqrt{46}$$

Length of projection, $AF = \left| \overrightarrow{AB} \bullet \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right|$

$$= \frac{1}{\sqrt{6}} \left| \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| = \frac{11}{\sqrt{6}}$$

Perpendicular distance $= \sqrt{46 - \left(\frac{11}{\sqrt{6}}\right)^2}$

$$= \sqrt{\frac{155}{6}}$$ **(ans)**

12.

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 10 \\ 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = \tfrac{1}{2}\overrightarrow{BC}$$

Since $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and B is a common point, A , B and C are collinear. **(ans)**

(b) Length of projection of $\overrightarrow{OA}$ on line OB

$$= \frac{1}{\sqrt{4^2 + 3^2 + 5^2}} \left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \right| = \frac{25}{5\sqrt{2}}$$

$= \frac{5\sqrt{2}}{2}$ units **(ans)**

13.

(a) $\overrightarrow{SU} = \mathbf{u} - \mathbf{s}$

$\overrightarrow{TU} = \mathbf{u} - \mathbf{t}$

$$\overrightarrow{SU} \bullet \overrightarrow{TU} = \mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{t} - \mathbf{s} \bullet \mathbf{u} + \mathbf{s} \bullet \mathbf{t}$$

$$= |\mathbf{u}|^2 - \mathbf{u} \bullet \mathbf{t} - (-\mathbf{t}) \bullet \mathbf{u} + |\mathbf{s}| \bullet |\mathbf{t}| \cos 180^\circ$$

$$= |\mathbf{u}|^2 - \mathbf{u} \bullet \mathbf{t} + \mathbf{u} \bullet \mathbf{t} + |\mathbf{u}| \bullet |\mathbf{u}|(-1)$$

$$= 0$$

Thus, SU and TU are perpendicular.

Alternatively,

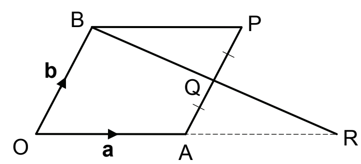
$$\overrightarrow{SU} \bullet \overrightarrow{TU} = (\mathbf{u} - \mathbf{s}) \bullet (\mathbf{u} - \mathbf{t})$$

$$= (\mathbf{u} - \mathbf{s}) \bullet (\mathbf{u} + \mathbf{s})$$

$= |\mathbf{u}|^2 - |\mathbf{s}|^2 = 0$ **(ans)**

(b)

(i)



$\overrightarrow{OP} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{OQ} = \tfrac{1}{2}\left[\left(\mathbf{a} + \mathbf{b}\right) + \mathbf{a}\right] = \tfrac{1}{2}\left(2\mathbf{a} + \mathbf{b}\right)$ **(ans)**

(ii) $\overrightarrow{OQ} = \tfrac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OR}\right)$

$\overrightarrow{OR} = 2\overrightarrow{OQ} - \overrightarrow{OB}$

$\overrightarrow{OR} = (2)\tfrac{1}{2}\left(2\mathbf{a} + \mathbf{b}\right) - \mathbf{b} = 2\mathbf{a}$

$\overrightarrow{OR} = 2\overrightarrow{OA}$

Since O, R and A are collinear, R is on OA produced.

Alternatively, BQ produced such that $BQ = QR$.

$\overrightarrow{BR} = 2\overrightarrow{BQ}$

$\overrightarrow{OR} = 2\overrightarrow{OA}$ **(ans)**

3.2 The scalar & vector products of vectors

Solutions

1.

(a) $\overrightarrow{OA} = \begin{pmatrix} c \\ 3 \\ d \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 10 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 10-c \\ 9 \\ 2-d \end{pmatrix}$

Let $\begin{pmatrix} 10-c \\ 9 \\ 2-d \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

$3\lambda = 9$

$\lambda = 3$

$10 - c = 3$

$2 - d = 0$

$c = 7$ and $d = 2$ **(ans)**

(b) Given: $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$.

$\overrightarrow{OP} \bullet \overrightarrow{AB} = 0$

$\overrightarrow{OP} \bullet \overrightarrow{AB} = \begin{pmatrix} 10+\lambda_1 \\ 12+3\lambda_1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 9 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 10+\lambda_1 \\ 12+3\lambda_1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 9 \\ 0 \end{pmatrix} = 0$

$30 + 3\lambda_1 + 108 + 27\lambda_1 = 0$

$30\lambda_1 = -138$

$\lambda_1 = -\frac{138}{30}$

$\overrightarrow{OP} = \begin{pmatrix} \frac{27}{5} \\ -\frac{9}{5} \\ 2 \end{pmatrix}$

Shortest $\left|\overrightarrow{OP}\right|$

$= \sqrt{\left(\frac{27}{5}\right)^2 + \left(\frac{9}{5}\right)^2 + 2^2} = 6.0332$ **(ans)**

(c) Let $\overrightarrow{OQ} = \begin{pmatrix} 10+\mu \\ 12+3\mu \\ 2 \end{pmatrix}$

Given: $\angle AOQ = \angle BOQ$.

$$\frac{\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 10+\mu \\ 12+3\mu \\ 2 \end{pmatrix}}{\sqrt{62}} = \frac{\begin{pmatrix} 10 \\ 12 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 10+\mu \\ 12+3\mu \\ 2 \end{pmatrix}}{\sqrt{248}}$$

$70 + 7\mu + 36 + 9\mu + 4$

$= \left(\frac{1}{2}\right)\left(100 + 10\mu + 144 + 36\mu + 4\right)$

$16\mu + 110 = 23\mu + 124$

$7\mu = -14$

$\mu = -2$

$\overrightarrow{OQ} = \begin{pmatrix} 10+(-2) \\ 12+3(-2) \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}$ **(ans)**

2.

(a) Let $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3+2\lambda \\ -1+\lambda \end{pmatrix}$.

$\overrightarrow{OA} \perp l$:

$\begin{pmatrix} 2 \\ 3+2\lambda \\ -1+\lambda \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0$

$6 + 4\lambda + (-1+\lambda) = 0$

$5\lambda = -5$

$\lambda = -1$

$A(2,1,-2)$ **(ans)**

(b) Let $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

$2 = -4 + 3\mu$ – Equation 1

$3 + 2\lambda = 3 - \mu$ – Equation 2

$-1 + \lambda = -4 + \mu$ – Equation 3

Solving Equations 2 and 3,

$\lambda = -1$ and $\mu = 2$

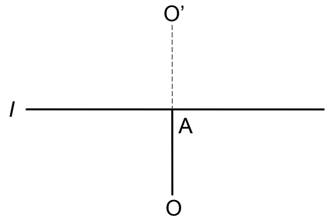
Substituting $\lambda = -1$ and $\mu = 2$ into Equation 1,

$LHS = 2$

$RHS = -4 + 6 = 2$

Hence, l and m intersect. **(ans)**

(c)



Let O' be the point of reflection of O about the line l.

$$\overrightarrow{OO'} = 2\overrightarrow{OA} = 2\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}4\\2\\-4\end{pmatrix} \quad \textbf{(ans)}$$

3.

(a) Line AB:

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})$, $\lambda \in \Re$

Line l:

$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\mu \in \Re$

Let

$$\begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}2\\4\\-5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$1 + \lambda = 2 + 2\mu$ – Equation 1

$2 - \lambda = 4 + \mu$ – Equation 2

$-1 + 5\lambda = -5 + \mu$ – Equation 3

From Equations 1 and 2,

$3 = 6 + 3\mu$

$\mu = -1$

From Equation 1,

$1 + \lambda = 2 + 2(-1)$

$\lambda = -1$

Substituting $\lambda = -1$ and $\mu = -1$ into Equation 3,

$LHS = -1 - 5 = -6$

$RHS = -5 - 1 = -6$

Since $LHS = RHS$, line AB and l intersect and the position vector of their point of intersection is:

$$\overrightarrow{OP} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}0\\3\\-6\end{pmatrix} \quad \textbf{(ans)}$$

(b) When $y = 4$, that is, when $y - 4 = 0$,

$\frac{x-2}{2} = 0 = z + 5$

$x = 2$, $z = -5$

$N(2,4,-5)$

$$\overrightarrow{AN} = \begin{pmatrix}2\\4\\-5\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\-4\end{pmatrix}$$

Direction vector of l:

$\mathbf{d} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\overrightarrow{AN} \cdot \mathbf{d} = \begin{pmatrix}1\\2\\-4\end{pmatrix} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 2 + 2 - 4 = 0$$

$\overrightarrow{AN} \perp l$ (shown)

Note that point N is the foot of the perpendicular from point A to line l.

Let A' be the reflection point of A about l.

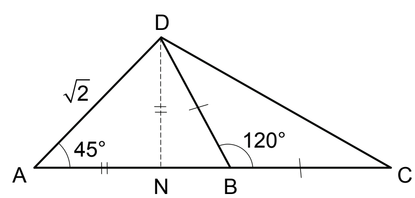
$\overrightarrow{PA'} = \overrightarrow{PA} + 2\overrightarrow{AN} = 3(\mathbf{i} + \mathbf{j} - \mathbf{k})$

Equation of image, by reflection, of the line AB in line l is given by:

$\mathbf{r} = 3\mathbf{i} - 6\mathbf{j} + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$, $t \in \Re$ **(ans)**

4.

(a)



Let N be the foot of the perpendicular from D to BC.

$AN = DN = \sqrt{2}\cos 45° = 1$

$$BD = \frac{DN}{\sin 60°} = \frac{2\sqrt{3}}{3}$$

$$AB = AN + NB = 1 + \frac{1}{\tan 60°} = 1 + \frac{\sqrt{3}}{3} \quad \textbf{(ans)}$$

(b) Area of ΔAOC

$$= \tfrac{1}{2}(DN)(AC) = \tfrac{1}{2}(DN)(AN + NB + BC)$$

$$= \tfrac{1}{2}(1)\left[1 + \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}\right] = \tfrac{1}{2}\left[1 + \sqrt{3}\right] \quad \textbf{(ans)}$$

5.

(a) $\overrightarrow{AB} = k\overrightarrow{BC}$

$$\begin{pmatrix} 4-\alpha \\ -1 \\ -2 \end{pmatrix} = k\begin{pmatrix} -5 \\ -5 \\ \beta \end{pmatrix}$$

$k = \frac{1}{5}$, $\alpha = 5$, $\beta = -10$ **(ans)**

(b) $$\cos\theta = \frac{\overrightarrow{OA} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\left|\overrightarrow{OA}\right| \left|\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right|} = \frac{\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{30}\sqrt{6}} = \frac{8}{\sqrt{180}}$$

$\theta = 53.4^\circ$ (to 1 decimal place)

Distance $= \left|\overrightarrow{OA}\right| \sin\theta = 4.40$ units (to 3 sig. fig.)

(ans)

(c) $\overrightarrow{OP} = \frac{1}{3}\overrightarrow{PB}$

$$\overrightarrow{OP} = \tfrac{1}{4}\overrightarrow{OB} = \tfrac{1}{4}\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ -2+\lambda \\ 2\lambda \end{pmatrix}, \ \lambda \in \Re$$

$$\overrightarrow{PQ} = \begin{pmatrix} 4+\lambda-1 \\ -2+\lambda+\frac{1}{2} \\ 2\lambda \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ -\frac{3}{2}+\lambda \\ 2\lambda \end{pmatrix}$$

$$\sqrt{5} = \frac{\left|\overrightarrow{PQ} \bullet \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|} = \frac{\left|\begin{pmatrix} 3+\lambda \\ -\frac{3}{2}+\lambda \\ 2\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|}{\sqrt{5}}$$

$6 + 2\lambda + \frac{3}{2} - \lambda = 5$

$\lambda = -\frac{5}{2}$

$$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} - \tfrac{5}{2}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 3 \\ -9 \\ -10 \end{pmatrix}$$ **(ans)**

6.

(a) $$\cos\angle CBD = \cos\left[\pi - \left(\frac{\pi}{3} + x\right)\right]$$

$$= -\cos\left(\frac{\pi}{3} + x\right) = -\left[\tfrac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right]$$

$$\approx \frac{\sqrt{3}}{2}x - \tfrac{1}{2}\left(1 - \frac{x^2}{2}\right) = \frac{x^2 + 2\sqrt{3}x - 2}{4}$$ **(ans)**

(b) $$\frac{a}{\sin\frac{\pi}{3}} = \frac{b}{\sin\left(\frac{\pi}{3} + x\right)} = \frac{c}{\sin\left(\frac{\pi}{3} - x\right)}$$

$$b = \frac{2a\sin\left(\frac{\pi}{3} + x\right)}{\sqrt{3}}$$

$$c = \frac{2a\sin\left(\frac{\pi}{3} - x\right)}{\sqrt{3}}$$

$b + c$

$$= \frac{2a}{\sqrt{3}}\left[\left(\sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x\right)\right.$$

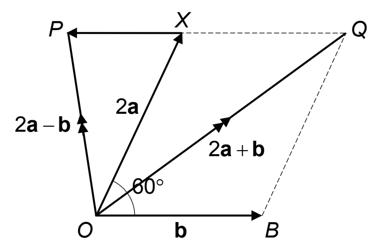
$$\left. + \left(\sin\frac{\pi}{3}\cos x - \cos\frac{\pi}{3}\sin x\right)\right]$$

$$= \frac{2a}{\sqrt{3}}\left(\sqrt{3}\cos x\right) \approx 2a\left(1 - \frac{x^2}{2}\right)$$

$b + c \approx a\left(2 - x^2\right)$ **(ans)**

7.

(a)



$|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$

$OX = 4$, $OB = 1$

$\angle OXP = \angle XOB = 60^\circ$ (alternate angles)

Using cosine rule for the triangle OAP,

$OP^2 = OB^2 + OX^2 - 2(OB)(OX)\cos 60^\circ$

$|2\mathbf{a} - \mathbf{b}|^2 = 1 + 16 - 2(1)(4)\left(\tfrac{1}{2}\right)$

$|2\mathbf{a} - \mathbf{b}| = \sqrt{13}$ (shown)

Using cosine rule for the triangle QOB,

$OQ^2 = OB^2 + BQ^2 - 2(OB)(BQ)\cos 120°$

$|2\mathbf{a}+\mathbf{b}|^2 = 1 + 16 - 2(1)(4)\left(-\frac{1}{2}\right)$

$|2\mathbf{a}+\mathbf{b}| = \sqrt{21}$ (shown)

$(2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}+\mathbf{b}) = |2\mathbf{a}-\mathbf{b}||2\mathbf{a}+\mathbf{b}|\cos\theta$

$4|\mathbf{a}|^2 - |\mathbf{b}|^2 = \sqrt{13}\sqrt{21}\cos\theta$

$\cos\theta = \dfrac{16-1}{\sqrt{13}\sqrt{21}}$

$\theta = 24.8°$

Alternatively,

$|2\mathbf{a}-\mathbf{b}|^2 = (2\mathbf{a}-\mathbf{b})\bullet(2\mathbf{a}-\mathbf{b})$

$= 4|\mathbf{a}|^2 - 4\mathbf{a}\bullet\mathbf{b} + |\mathbf{b}|^2$

$= 4(4) - 4(2)(1)\cos 60° + 1 = 13$

$|2\mathbf{a}-\mathbf{b}| = \sqrt{13}$ (shown)

$|2\mathbf{a}+\mathbf{b}|^2 = (2\mathbf{a}+\mathbf{b})\bullet(2\mathbf{a}+\mathbf{b})$

$= 4|\mathbf{a}|^2 + 4\mathbf{a}\bullet\mathbf{b} + |\mathbf{b}|^2$

$= 4(4) + 4(2)(1)\cos 60° + 1 = 21$

$|2\mathbf{a}+\mathbf{b}| = \sqrt{21}$ (shown) **(ans)**

(b)

(i) Vector equation of the line AB:

$\mathbf{r} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\3\end{pmatrix}, \ \lambda \in \Re$ **(ans)**

(ii) P is a point on the line AB.

$\overrightarrow{OP} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\3\end{pmatrix}$ for some $\lambda \in \Re$

$OP = 13$

$\sqrt{3^2 + (1+\lambda)^2 + (3+3\lambda)^2} = 13$

$9 + 1 + 2\lambda + \lambda^2 + 9 + 18\lambda^2 + 9\lambda^2 = 169$

$\lambda^2 + 2\lambda - 15 = 0$

$(\lambda+5)(\lambda-3) = 0$

$\lambda = -5$ or 3

Hence,

$\overrightarrow{OP_1} = \begin{pmatrix}3\\1\\3\end{pmatrix} + (-5)\begin{pmatrix}0\\1\\3\end{pmatrix}$

$\overrightarrow{OP_2} = \begin{pmatrix}3\\1\\3\end{pmatrix} + (3)\begin{pmatrix}0\\1\\3\end{pmatrix}$

$\left|\overrightarrow{P_1P_2}\right| = 8\sqrt{0+1^2+3^2} = 8\sqrt{10}$ **(ans)**

8.

(a) $\overrightarrow{AB} = \begin{pmatrix}5\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}3\\-5\\4\end{pmatrix}$

Equation of line AB:

$\mathbf{r} = (2\mathbf{i}+3\mathbf{j}-\mathbf{k}) + \lambda(3\mathbf{i}-5\mathbf{j}+4\mathbf{k})$ **(ans)**

(b) When line AB meets the xy-plane, $z = 0$.

$-1 + 4\lambda = 0$

$\lambda = \frac{1}{4}$

$\overrightarrow{OD} = (2\mathbf{i}+3\mathbf{j}-\mathbf{k}) + \frac{1}{4}(3\mathbf{i}-5\mathbf{j}+4\mathbf{k})$

$= \frac{11}{4}\mathbf{i} + \frac{7}{4}\mathbf{j}$

Coordinates of D: $\left(\frac{11}{4}, \frac{7}{4}, 0\right)$ **(ans)**

(c) $\overrightarrow{AC} = \begin{pmatrix}4\\1\\-2\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$

Length of projection, k

$= \dfrac{\left|\overrightarrow{AC}\bullet\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \dfrac{\left|\begin{pmatrix}2\\-2\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\-5\\4\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\-5\\4\end{pmatrix}\right|} = \dfrac{|6+10-4|}{\sqrt{9+25+16}}$

$= \dfrac{12}{\sqrt{50}} = \frac{6}{5}\sqrt{2}$ **(ans)**

(d) Equation of line AC:

$\mathbf{r} = (2\mathbf{i}+3\mathbf{j}-\mathbf{k}) + \mu(2\mathbf{i}-2\mathbf{j}-\mathbf{k})$

P lies on the line AC.

$\overrightarrow{OP} = \begin{pmatrix}2+2\mu\\3-2\mu\\-1-\mu\end{pmatrix}$

$$\overrightarrow{DP}=\overrightarrow{OP}-\overrightarrow{OD}=\begin{pmatrix}2+2\mu\\3-2\mu\\-1-\mu\end{pmatrix}-\begin{pmatrix}\frac{11}{4}\\\frac{7}{4}\\0\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{3}{4}+2\mu\\\frac{5}{4}-2\mu\\-1-\mu\end{pmatrix}$$

$\overrightarrow{DP}$ is perpendicular to the line AC.

$$\overrightarrow{DP}\bullet\begin{pmatrix}2\\-2\\-1\end{pmatrix}=0$$

$$2\left(-\tfrac{3}{4}+2\mu\right)-2\left(\tfrac{5}{4}-2\mu\right)-(-1-\mu)=0$$

$$\mu=\tfrac{1}{3}$$

$$\overrightarrow{OP}=\begin{pmatrix}2+2\mu\\3-2\mu\\-1-\mu\end{pmatrix}=\tfrac{8}{3}\mathbf{i}+\tfrac{7}{3}\mathbf{j}-\tfrac{4}{3}\mathbf{k}\quad\textbf{(ans)}$$

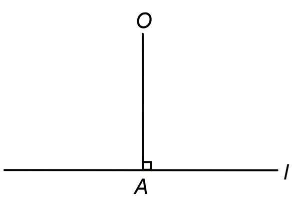
9.

(a) Cartesian equation:

$$\frac{x-1}{-1}=\frac{y-3}{1}, z=4\quad\textbf{(ans)}$$

(b) Since A lies on the line l,

$$\overrightarrow{OA}=\begin{pmatrix}1-\lambda\\3+\lambda\\4\end{pmatrix}\text{ for some fixed }\lambda\in\Re$$



$$\overrightarrow{OA}\bullet\begin{pmatrix}-1\\1\\0\end{pmatrix}=0$$

$$-(1-\lambda)+(3+\lambda)=0$$

$$2\lambda+2=0$$

$$\lambda=-1$$

$$\overrightarrow{OA}=\begin{pmatrix}2\\2\\4\end{pmatrix}\quad\textbf{(ans)}$$

(c) Since B lies on the line l,

$$\overrightarrow{OB}=\begin{pmatrix}1-\lambda\\3+\lambda\\4\end{pmatrix}\text{ for some fixed }\lambda\in\Re$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1-\lambda\\3+\lambda\\4\end{pmatrix}-\begin{pmatrix}2\\2\\4\end{pmatrix}=\begin{pmatrix}-1-\lambda\\1+\lambda\\0\end{pmatrix}$$

Given: $\left|\overrightarrow{AB}\right|=2\sqrt{2}$,

$$(-1-\lambda)^2+(1+\lambda)^2=8$$

$$(1+\lambda)^2=4$$

$$\lambda^2+2\lambda-3=0$$

$$(\lambda+3)(\lambda-1)=0$$

$$\lambda=-3\text{ or }\lambda=1$$

$$\overrightarrow{OB}=\begin{pmatrix}0\\4\\4\end{pmatrix}\text{ or }\overrightarrow{OB}=\begin{pmatrix}4\\0\\4\end{pmatrix}\quad\textbf{(ans)}$$

(d)
$$\cos\frac{\pi}{4}=\frac{\left|\begin{pmatrix}a\\1\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\0\end{pmatrix}\right|}{\sqrt{2(a^2+2)}}=\frac{|-a+1|}{\sqrt{2(a^2+2)}}=\frac{1}{\sqrt{2}}$$

$$|-a+1|=\sqrt{a^2+2}$$

$$(-a+1)^2=a^2+2$$

$$a^2-2a+1=a^2+2$$

$$-2a=1$$

$$a=-\tfrac{1}{2}\quad\textbf{(ans)}$$

10.

(a) $-6+5=\dfrac{2+1}{-3}=\dfrac{3-7}{4}=-1$ **(ans)**

(b)
$$\begin{pmatrix}-1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-5\\-1\\7\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\4\end{pmatrix}$$

$-1+2\lambda=-5+\mu$ – Equation 1

$2-\lambda=-1-3\mu$ – Equation 2

$2+\lambda=7+4\mu$ – Equation 3

Solving Equations 2 and 3,

$\lambda=-3$, $\mu=-2$

Substituting into Equation 1,

$LHS=-7=RHS$

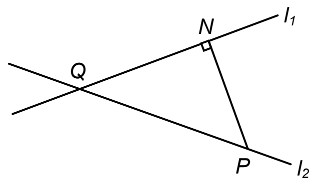
Hence, l_1 and l_2 intersect.

Coordinates of point of intersection:

$(-7,5,-1)$ **(ans)**

(c) $\theta = \cos^{-1}\dfrac{\begin{pmatrix}2\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\4\end{pmatrix}}{\sqrt{6}\sqrt{26}} = \cos^{-1}\dfrac{9}{\sqrt{156}}$

$= 43.9°$



$PQ = \left|\begin{pmatrix}-7\\5\\-1\end{pmatrix} - \begin{pmatrix}-6\\2\\3\end{pmatrix}\right| = \left|\begin{pmatrix}-1\\3\\-4\end{pmatrix}\right| = \sqrt{1+9+16}$

$= \sqrt{26}$

$PN = \sqrt{26}\sin 43.9° = 3.54$ **(ans)**

11.

(a) Equation of line AD:

$\mathbf{r} = \begin{pmatrix}2\\8\\7\end{pmatrix} + \lambda\begin{pmatrix}-3\\-5\\2\end{pmatrix},\ \lambda \in \Re$ **(ans)**

(b) Let E be the foot of the perpendicular from O to line AD.

$\overrightarrow{OE} = \begin{pmatrix}2-3\lambda\\8-5\lambda\\7+2\lambda\end{pmatrix}$

$\begin{pmatrix}2-3\lambda\\8-5\lambda\\7+2\lambda\end{pmatrix}\bullet\begin{pmatrix}-3\\-5\\2\end{pmatrix} = 0$

$\lambda = \frac{16}{19}$

$\overrightarrow{OE} = \frac{1}{19}\begin{pmatrix}-10\\72\\165\end{pmatrix}$ **(ans)**

(c) $\begin{pmatrix}2-3\lambda\\8-5\lambda\\7+2\lambda\end{pmatrix} = \mu\begin{pmatrix}1\\-3\\10\end{pmatrix}$

$2-3\lambda = \mu$ – Equation 1

$8-5\lambda = -3\mu$ – Equation 2

$7+2\lambda = 10\mu$ – Equation 3

Solving Equations 1 and 2,

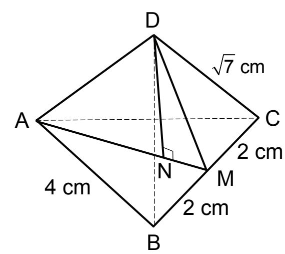
$\lambda = 1$

$\mu = -1$

This pair of solution does not satisfy Equation 3. Hence, the two lines do not intersect and are skew lines. **(ans)**

12.

(a)



Angle between planes ABC and BCD

$= \angle AMD = 60°$

$DM = \sqrt{7-2^2} = \sqrt{3}$

$AM = \sqrt{4^2-2^2} = \sqrt{12}$

$AD^2 = AM^2 + DM^2 - 2(AM)(DM)\cos 60°$

$= 12 + 3 - 2\left(\sqrt{3}\right)\left(\sqrt{12}\right)\left(\tfrac{1}{2}\right)$

$AD = \sqrt{9} = 3$ cm **(ans)**

(b) Angle between BD and plane ABC

$= \angle DBN$

$DN = \sqrt{3}\sin 60° = \frac{3}{2}$

$\angle DBN = \sin^{-1}\left(\dfrac{DN}{DB}\right) = \sin^{-1}\left(\dfrac{3}{2\sqrt{7}}\right) = 34.5°$

(to the nearest $0.1°$) **(ans)**

13.

(a) $\overrightarrow{OA}=\begin{pmatrix}5\\-2\\4\end{pmatrix}$

$\overrightarrow{OB}=\begin{pmatrix}1\\0\\2\end{pmatrix}$

$l_L:\mathbf{r}=\begin{pmatrix}5\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix},\ \lambda\in\Re$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-4\\2\\-2\end{pmatrix}$

Vector equation of line passing through A and B:

$\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix},\ \mu\in\Re$ **(ans)**

(b) $\overrightarrow{ON}=\begin{pmatrix}5\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix}$ for some $\lambda\in\Re$.

Since line BN is perpendicular to line L,

$\overrightarrow{BN}\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}=0$

$\left[\begin{pmatrix}5+\lambda\\-2-3\lambda\\4+2\lambda\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}\right]\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}=0$

$4+6+4+\lambda(1+9+4)=0$

$\lambda=-1$

$\overrightarrow{ON}=\begin{pmatrix}5\\-2\\4\end{pmatrix}-\begin{pmatrix}1\\-3\\2\end{pmatrix}=\begin{pmatrix}4\\1\\2\end{pmatrix}$

$\overrightarrow{ON}=\frac{1}{2}\left[\overrightarrow{OB}+\overrightarrow{OD}\right]$

$\overrightarrow{OD}=2\overrightarrow{ON}-\overrightarrow{OB}=2\begin{pmatrix}4\\1\\2\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}7\\2\\2\end{pmatrix}$ **(ans)**

(c) $\overrightarrow{AN}=\begin{pmatrix}4\\1\\2\end{pmatrix}-\begin{pmatrix}5\\-2\\4\end{pmatrix}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}$

Length of projection of $\overrightarrow{AB}$ onto L

$=AN=\sqrt{(-1)^2+3^2+(-2)^2}=\sqrt{14}$ units

Alternatively,

Length of projection of $\overrightarrow{AB}$ onto L

$=\dfrac{\left|\overrightarrow{AB}\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|}{\sqrt{1^2+(-3)^2+2^2}}=\dfrac{\left|\begin{pmatrix}-4\\2\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|}{\sqrt{14}}$

$=\dfrac{|-4-6-4|}{\sqrt{14}}=\sqrt{14}$ units **(ans)**

(d) $\overrightarrow{OQ}=\begin{pmatrix}1+2\mu\\-\mu\\2+\mu\end{pmatrix}$ for some $\mu\in\Re$

$\cos\angle AOQ=\cos\angle BOQ$

$\dfrac{\overrightarrow{OA}\bullet\overrightarrow{OQ}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OQ}\right|}=\dfrac{\overrightarrow{OB}\bullet\overrightarrow{OQ}}{\left|\overrightarrow{OB}\right|\left|\overrightarrow{OQ}\right|}$

$\dfrac{\begin{pmatrix}5\\-2\\4\end{pmatrix}\bullet\begin{pmatrix}1+2\mu\\-\mu\\2+\mu\end{pmatrix}}{\sqrt{5^2+(-2)^2+4^2}}=\dfrac{\begin{pmatrix}1\\0\\2\end{pmatrix}\bullet\begin{pmatrix}1+2\mu\\-\mu\\2+\mu\end{pmatrix}}{\sqrt{1^2+2^2}}$

$\dfrac{5+8+\mu(10+2+4)}{\sqrt{45}}=\dfrac{1+4+\mu(2+2)}{\sqrt{5}}$

$\dfrac{13+16\mu}{3\sqrt{5}}=\dfrac{5+4\mu}{\sqrt{5}}$

$13+16\mu=3(5+4\mu)$

$\mu=\frac{1}{2}$

$\overrightarrow{OQ}=\begin{pmatrix}2\\-\frac{1}{2}\\\frac{5}{2}\end{pmatrix}$ **(ans)**

14.

(a) Vector equation of L_1:

$$\mathbf{r}=\begin{pmatrix}2\\2\\\frac{3}{2}\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\2\end{pmatrix},\ \mu\in\Re$$

Cosine of the acute angle between the two lines:

$$=\frac{\begin{pmatrix}-3\\0\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\-1\end{pmatrix}}{\sqrt{13}\sqrt{6}}=\frac{1}{\sqrt{78}}\ \text{(shown)}\ \textbf{(ans)}$$

(b) Equating the two lines,

$2-3\mu=1-\lambda$ – Equation 1

$2=1+2\lambda$ – Equation 2

$\frac{3}{2}+2\mu=3-\lambda$ – Equation 3

From Equation 2,

$\lambda=\frac{1}{2}$

From Equation 1,

$2-3\mu=1-\frac{1}{2}$

$\mu=\frac{1}{2}$

Checking, from Equation 3,

$LHS=\frac{3}{2}+1=\frac{5}{2}$

$RHS=3-\frac{1}{2}=LHS$

Position vector of the point of intersection between L_1 and L_2:

$$\begin{pmatrix}1-\frac{1}{2}\\1+1\\3-\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\2\\\frac{5}{2}\end{pmatrix}\ \textbf{(ans)}$$

15.

(a) $\overrightarrow{OB}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OX}\right)$

$$\overrightarrow{OX}=2\overrightarrow{OB}-\overrightarrow{OA}=2\begin{pmatrix}2\\0\\-2\end{pmatrix}-\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}5\\-4\\-7\end{pmatrix}\ \textbf{(ans)}$$

(b) $$\overrightarrow{AB}=\begin{pmatrix}2\\0\\-2\end{pmatrix}-\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}3\\-4\\-5\end{pmatrix}$$

Length of projection of $\overrightarrow{OX}$ onto the line AB

$$=\left|\overrightarrow{OX}\bullet\frac{\overrightarrow{AB}}{AB}\right|=\left|\frac{\begin{pmatrix}5\\-4\\-7\end{pmatrix}\bullet\begin{pmatrix}3\\-4\\-5\end{pmatrix}}{\sqrt{9+16+25}}\right|=\frac{66}{5\sqrt{2}}$$

$$\left|\overrightarrow{OX}\right|=\sqrt{25+16+49}=\sqrt{90}$$

Perpendicular distance of O from the line AB

$$=\sqrt{90-\left(\frac{33\sqrt{2}}{5}\right)^2}=\frac{6\sqrt{2}}{5}\ \textbf{(ans)}$$

(c) Vector equation of the line AB:

$$\mathbf{r}=\begin{pmatrix}-1\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix},\ \lambda\in\Re$$

Let N be the foot of the perpendicular from C to the line AB.

Since N lies on AB,

$$\overrightarrow{ON}=\begin{pmatrix}-1+3\lambda\\4-4\lambda\\3-5\lambda\end{pmatrix}\text{ for some }\lambda\in\Re$$

$$\overrightarrow{CN}\bullet\begin{pmatrix}3\\-4\\-5\end{pmatrix}=0$$

$$\begin{pmatrix}-1+3\lambda+6\\4-4\lambda+1\\3-5\lambda-9\end{pmatrix}\bullet\begin{pmatrix}3\\-4\\-5\end{pmatrix}=0$$

$$\begin{pmatrix}3\lambda+5\\5-4\lambda\\-6-5\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\-4\\-5\end{pmatrix}=0$$

$9\lambda+15-20+16\lambda+30+25\lambda=0$

$50\lambda=-25$

$\lambda=-\frac{1}{2}$

$$\overrightarrow{ON}=\begin{pmatrix}-1-\frac{3}{2}\\4+2\\3+\frac{5}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-5\\12\\11\end{pmatrix}\ \textbf{(ans)}$$

16.

(a) $\overrightarrow{OA} = 4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}$

$\overrightarrow{OB} = 6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{OC} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}) - (4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k})$

$= 2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$

Equation of line L:

$\mathbf{r} = (4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}) + \lambda(2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})$, $\lambda \in \Re$

(ans)

(b) $\overrightarrow{AC} = (\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}) - (4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}) = -3\mathbf{i} - 3\mathbf{k}$

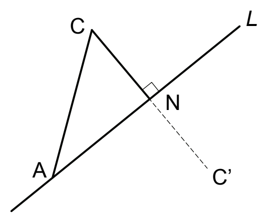
Length of projection of AC on L

$$= \left|\overrightarrow{AC} \bullet \overrightarrow{AB}\right|$$

$$= \left|\frac{(-3\mathbf{i} - 3\mathbf{k}) \bullet (2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})}{\sqrt{4+4+64}}\right| = \left|\frac{-6+24}{\sqrt{72}}\right|$$

$$= \frac{18}{6\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad \textbf{(ans)}$$

(c)



$$\overrightarrow{AN} = \left|\overrightarrow{AN}\right| \bullet (2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})\frac{1}{\sqrt{72}}$$

$$= \frac{3\sqrt{2}}{2}\left(\frac{1}{6\sqrt{2}}\right)(2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})$$

$$= \tfrac{1}{2}(\mathbf{i} - \mathbf{j} - 4\mathbf{k})$$

$$\overrightarrow{CC'} = 2\overrightarrow{CN}$$

$$\overrightarrow{OC'} = 2\overrightarrow{ON} - \overrightarrow{OC} = 2\left(\overrightarrow{AN} + \overrightarrow{OA}\right) - \overrightarrow{OC}$$

$$= 2\left[\tfrac{1}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{j} - 2\mathbf{k} + 4\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}\right]$$

$$-\left(\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}\right)$$

$$= 8\mathbf{i} + 9\mathbf{j} + 5\mathbf{k} \quad \textbf{(ans)}$$

17.

(a) Since L_1 and L_2 intersect,

$1 + \lambda(1) = -1$

$\lambda = -2$

Position vector: $\begin{pmatrix} -5 \\ 5 \\ -1 \end{pmatrix}$ **(ans)**

(b) $L_2 : m_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

Let θ be the acute angle between L_1 and L_2.

$$\cos\theta = \left|\frac{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}}{\sqrt{14}\sqrt{13}}\right| = 0.96362411$$

$\theta = 15.5°$ **(ans)**

18. $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$

$l : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\lambda \in \Re$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

Let θ be the angle between $\overrightarrow{AB}$ and l.

$$\cos\theta = \frac{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{\sqrt{6}}{3}$$

Length of projection of $\overrightarrow{AB}$ on l

$$= \left|\overrightarrow{AB}\right|\cos\theta = \sqrt{2^2 + 2^2 + 2^2}\sqrt{\tfrac{2}{3}}$$

$$= 2\sqrt{2} \quad \textbf{(ans)}$$

19.

(a) Let P be on l_1 such that $\overrightarrow{AP} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$.

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} \lambda \\ -2+\lambda \\ 3\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda-1 \\ -4+\lambda \\ 3\lambda-2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda-1 \\ \lambda-4 \\ 3\lambda-2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$$

$(\lambda-1)+(\lambda-4)+3(3\lambda-2)=0$

$\lambda-1+\lambda-4+9\lambda-6=0$

$11\lambda=11$

$\lambda=1$

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2+1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \textbf{(ans)}$$

(b) $l_1 : \mathbf{r} = \begin{pmatrix} \lambda \\ -2+\lambda \\ 3\lambda \end{pmatrix}$

$l_2 : \mathbf{r} = \begin{pmatrix} 1+2\mu \\ 2-2\mu \\ 2+\mu \end{pmatrix}$

If l_1 and l_2 intersect,

$$\begin{pmatrix} \lambda \\ -2+\lambda \\ 3\lambda \end{pmatrix} = \begin{pmatrix} 1+2\mu \\ 2-2\mu \\ 2+\mu \end{pmatrix}$$

$\lambda = 1+2\mu$ – Equation 1

$-2+\lambda = 2-2\mu$ – Equation 2

$3\lambda = 2+\mu$ – Equation 3

Substituting Equation 1 into Equation 2,

$-2+(1+2\mu)=2-2\mu$

$4\mu=3$

$\mu=\frac{3}{4}$

$\lambda=\frac{5}{2}$

Substituting Equation 1 into Equation 3,

$3(1+2\mu)=2+\mu$

$3+6\mu=2+\mu$

$5\mu=-1$

$\mu=-\frac{1}{5}$

Since there is no unique λ and μ, the two lines l_1 and l_2 do not intersect. **(ans)**

20. $l : \mathbf{r} = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} + s\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}, \ s \in \Re$

Let the foot of the perpendicular from C to the above line be P.

$$\overrightarrow{CP} \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\left(\overrightarrow{OP} \bullet \overrightarrow{OC}\right) \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 5-2s \\ 6+3s \\ 4-s \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right] \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$s=-\frac{3}{14}$

$$\overrightarrow{OP} = \begin{pmatrix} \frac{38}{7} \\ \frac{75}{14} \\ \frac{53}{14} \end{pmatrix}$$

$$AP = \left|\overrightarrow{OP} - \overrightarrow{OA}\right| = \left|\begin{pmatrix} \frac{33}{7} \\ \frac{75}{14} \\ \frac{53}{14} \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} \frac{6}{14} \\ -\frac{9}{14} \\ -\frac{3}{14} \end{pmatrix}\right| = \frac{3\sqrt{14}}{14} \text{ units}$$

Alternatively,

Length of projection, AP

$$= \left|\overrightarrow{AC} \frac{\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}}{\sqrt{4+9+1}}\right| = \frac{3}{\sqrt{14}} \text{ units} \quad \textbf{(ans)}$$

21.

(a) $\overrightarrow{OA} \perp \overrightarrow{AB} = \overrightarrow{OA} \bullet \overrightarrow{AB} = 0$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}2-1\\3-2\\p-3\end{pmatrix} = 0$$

$1+2+3(p-3)=0$

$p=2$ (shown) **(ans)**

(b) $\overrightarrow{OC} = \overrightarrow{AB} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ **(ans)**

(c) $\overrightarrow{BC} = \begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}2\\3\\2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix}$

Alternatively,

$\overrightarrow{BC} = \overrightarrow{AO} = -\begin{pmatrix}1\\2\\3\end{pmatrix}$

Vector equation of line BC:

$\mathbf{r} = \begin{pmatrix}2\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}, \ \lambda \in \Re$ **(ans)**

(d) $l: \dfrac{x-1}{3} = \dfrac{y-1}{2}, z=1$

$\mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \mu\begin{pmatrix}3\\2\\0\end{pmatrix}, \ \mu \in \Re$

Equating BC and l,

$$\begin{pmatrix}3+\lambda\\3+2\lambda\\2+3\lambda\end{pmatrix} = \begin{pmatrix}1+3\mu\\1+2\mu\\1\end{pmatrix}$$

$1+\lambda = 3\mu$ – Equation 1

$2+2\lambda = 2\mu$ – Equation 2

$1+3\lambda = 0$ – Equation 3

Solving Equation 1 and Equation 2,

$\mu = 0$

$\lambda = -1$

However, Equation 3 gives $\lambda = -\frac{1}{3}$

Hence, the two lines do not intersect.

Furthermore, they are not parallel since $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ is not parallel to $3\mathbf{i}+2\mathbf{j}$. Hence, the two lines are skew. **(ans)**

22.

(a) $\overrightarrow{OA} = \begin{pmatrix}8\\3\\2\end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix}-2\\3\\4\end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-2\\3\\4\end{pmatrix} - \begin{pmatrix}8\\3\\2\end{pmatrix} = \begin{pmatrix}-10\\0\\2\end{pmatrix}$

$AB = \sqrt{100+4} = 2\sqrt{26}$ (shown) **(ans)**

(b) Vector equation of AB:

$\mathbf{r} = \begin{pmatrix}8\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-10\\0\\2\end{pmatrix}$

Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$.

Hence, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}8\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-10\\0\\2\end{pmatrix}$

$x = 8-10\lambda$

$\lambda = \dfrac{8-x}{10}$

$y=3$

$z = 2+2\lambda$

$\lambda = \dfrac{z-2}{2}$

Cartesian Equation of AB:

$\dfrac{8-x}{10} = \dfrac{z-2}{2}, y=3$ **(ans)**

(c) Point B passes through l.

Length of projection

$$= \frac{\left|\begin{pmatrix}-10\\0\\2\end{pmatrix} \bullet \begin{pmatrix}2\\6\\5\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\6\\5\end{pmatrix}\right|} = \frac{|-20+0+10|}{\sqrt{2^2+6^2+5^2}} = \frac{10}{\sqrt{65}}$$

(ans)

(d) $\begin{pmatrix}-10\\0\\2\end{pmatrix}\bullet\begin{pmatrix}2\\6\\5\end{pmatrix}=\left(\sqrt{104}\right)\left(\sqrt{65}\right)\cos\theta$

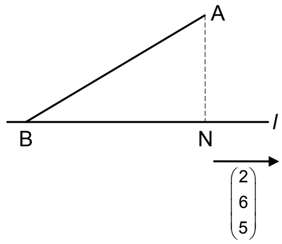
$\cos\theta=\dfrac{-20+0+10}{\sqrt{104}\sqrt{65}}$

$\theta=96.98^\circ$

Acute angle between AB and l

$=180^\circ-96.98^\circ=83^\circ$ **(ans)**

(e)



$\overrightarrow{ON}=\begin{pmatrix}-2\\3\\4\end{pmatrix}+t\begin{pmatrix}2\\6\\5\end{pmatrix}$ for some value of t

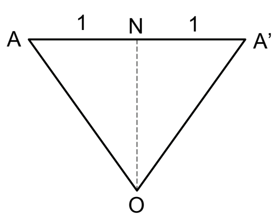
$\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}-10+2t\\6t\\2+5t\end{pmatrix}$

$\begin{pmatrix}-10+2t\\6t\\2+5t\end{pmatrix}\bullet\begin{pmatrix}2\\6\\5\end{pmatrix}=0$

$-20+4t+36t+10+25t=0$

$65t=10 \quad\Rightarrow\quad t=\frac{10}{65}=\frac{2}{13}$

$\overrightarrow{ON}=\begin{pmatrix}-2\\3\\4\end{pmatrix}+\frac{2}{13}\begin{pmatrix}2\\6\\5\end{pmatrix}=\frac{1}{13}\begin{pmatrix}-22\\51\\62\end{pmatrix}$



By Ratio Theorem,

$\overrightarrow{ON}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA}'}{2}$

$\overrightarrow{OA}'=2\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}-\frac{44}{13}\\\frac{102}{13}\\\frac{124}{13}\end{pmatrix}-\begin{pmatrix}8\\3\\2\end{pmatrix}=\begin{pmatrix}-\frac{148}{13}\\\frac{63}{13}\\\frac{98}{13}\end{pmatrix}$ **(ans)**

23.

(a) $\overrightarrow{OP}=\begin{pmatrix}3\\4\\1\end{pmatrix}+s\begin{pmatrix}-1\\2\\0\end{pmatrix}$ for some $s\in\Re$

Since OP is perpendicular to l,

$\overrightarrow{OP}\bullet\begin{pmatrix}-1\\2\\0\end{pmatrix}=0$

$\begin{pmatrix}3-s\\4+2s\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\0\end{pmatrix}=0$

$-3+s+8-4s=0$

$s=-1$

$\overrightarrow{OP}=\begin{pmatrix}3\\4\\1\end{pmatrix}+\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}4\\2\\1\end{pmatrix}$ **(ans)**

(b) Let $\begin{pmatrix}3\\4\\1\end{pmatrix}+s\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}6\\a\\0\end{pmatrix}+t\begin{pmatrix}1\\3\\a\end{pmatrix}$

$3-s=6+t$ – Equation 1

$4+2s=a+3t$ – Equation 2

$1=at$ – Equation 3

From Equation 1,

$s=-3-t$ – Equation 4

Substituting Equation 4 into Equation 2,

$4+2(-3-t)=a+3t$

$5t=-2-a$ – Equation 5

Substituting Equation 5 into Equation 3,

$5=a(-2-a)$

$a^2+2a+5=0$

$(a-1)^2+4=0$

Since there are no real solutions for the equation $(a+1)^2+4=0$, there does not exist real values of a such that the two lines l and m intersect.

Hence, the two lines do not have a common point.

(ans)

(c) $\cos 60^\circ = \left| \frac{\begin{pmatrix}1\\3\\a\end{pmatrix} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{10+a^2}(1)} \right|$

$\frac{1}{2} = \frac{|a|}{\sqrt{10+a^2}}$

$4a^2 = 10 + a^2$

$a = \pm\sqrt{\frac{10}{3}}$ **(ans)**

---

24. $\overrightarrow{AB} = \begin{pmatrix}-1\\1\\-2\end{pmatrix}$

$\overrightarrow{CD} = \begin{pmatrix}a-5\\-1\\b\end{pmatrix}$

Since AB and CD are perpendicular,

$\overrightarrow{AB} \bullet \overrightarrow{CD} = 0$

$-a + 5 - 1 - 2b = 0$

$a + 2b = 4$ – Equation 1

Since the lines AB and CD intersect,

$\begin{pmatrix}2\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix} = \begin{pmatrix}5\\2\\0\end{pmatrix} + \mu\begin{pmatrix}a-5\\-1\\b\end{pmatrix}$

$\begin{pmatrix}2-\lambda\\3+\lambda\\2-2\lambda\end{pmatrix} = \begin{pmatrix}5+\mu a-5\mu\\2-\mu\\\mu b\end{pmatrix}$

$2 - \lambda = 5 + \mu a - 5\mu$ – Equation 2

$\mu + \lambda = -1$ – Equation 3

$\mu b + 2\lambda = 2$ – Equation 4

From Equations 1 and 3,

$a = 4 - 2b$

$\mu = -1 - \lambda$

Substituting into Equations 2 and 4,

$2 - \lambda = 5 + \mu a - 5\mu$

$2 - \lambda = 5 + (-1-\lambda)(4-2b) - 5(-1-\lambda)$

$2 - \lambda = 5 - 4 + 2b - 4\lambda + 2b\lambda + 5 + 5\lambda$

$-4 = 2\lambda + 2b + 2b\lambda$

$b = \frac{-2-\lambda}{1+\lambda}$ – Equation 5

$\mu b + 2\lambda = 2$

$(-1-\lambda)b + 2\lambda = 2$

$\frac{-2+2\lambda}{1+\lambda} = b$ – Equation 6

Equating Equations 5 and 6,

$\frac{-2-\lambda}{1+\lambda} = \frac{-2+2\lambda}{1+\lambda}$

$-2 - \lambda = -2 + 2\lambda$

$\lambda = 0$

$b = -2$

$a = 4 - 2(-2) = 8$ (shown)

Point of intersection between AB and CD: $\begin{pmatrix}2\\3\\2\end{pmatrix}$

This point is coincidentally $\overrightarrow{OA}$.

Length of projection, $\left|\overrightarrow{AB}\right|$

$= \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$ **(ans)**

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25. Vector equation of line l_1:

$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ or

$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k} + s(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$

Vector equation of l_2:

$\mathbf{r} = \begin{pmatrix}-2\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}a\\b\\0\end{pmatrix}, \lambda \in \Re$

When the two lines intersect,

$\begin{pmatrix}2+s\\-1+s\\4-3s\end{pmatrix} = \begin{pmatrix}-2+\lambda a\\2+\lambda b\\2\end{pmatrix}$

$2 + s = -2 + \lambda a$ – Equation 1

$-1 + s = 2 + \lambda b$ – Equation 2

$4 - 3s = 2$ – Equation 3

or

When the two lines intersect,

$\begin{pmatrix}1+s\\-2+s\\7-3s\end{pmatrix} = \begin{pmatrix}-2+\lambda a\\2+\lambda b\\2\end{pmatrix}$

$1 + s = -2 + \lambda a$ – Equation 1

$-2 + s = 2 + \lambda b$ – Equation 2

$7 - 3s = 2$ – Equation 3

From Equation 3,

$s = \frac{2}{3}$ or $s = \frac{5}{3}$

Point of intersection:

$2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \frac{2}{3}(\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = \frac{8}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + 2\mathbf{k}$

Substituting $s = \frac{2}{3}$ into Equations 1 and 2,

$\lambda a = \frac{14}{3}$

$\lambda b = -\frac{7}{3}$

Thus, $a = -2b$

Direction vector of l_2 : $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\left| \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \right| = \left| \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \right| \cos\theta$$

$$\cos\theta = \frac{|-1|}{\sqrt{55}}$$

$\theta = 82.3°$ **(ans)**

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26. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ b-1 \\ 2 \end{pmatrix}$

Since l_1 and l_2 intersect at right angles,

$$\overrightarrow{AB} \bullet \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 \\ b-1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 3(b-1) - 12 = 0$$

$b = 5$

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \ t \in \Re$$

$$l_2 : \mathbf{r} = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \ \lambda \in \Re$$

Since l_1 and l_2 intersect,

$$\begin{pmatrix} 1+2t \\ -2+3t \\ 5-6t \end{pmatrix} = \begin{pmatrix} a \\ 1+4\lambda \\ -1+2\lambda \end{pmatrix}$$

$1 + 2t = a$ – Equation 1

$-2 + 3t = 1 + 4\lambda$ – Equation 2

$5 - 6t = -1 + 2\lambda$ – Equation 3

Solving,

$t = 1$

$a = 3$

P lies on l_1 ,

$$\overrightarrow{OP} = \begin{pmatrix} 1+2t \\ -2+3t \\ 5-6t \end{pmatrix}$$

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \begin{pmatrix} -2+2t \\ -7+3t \\ 4-6t \end{pmatrix}$$

$$\left|\overrightarrow{BP}\right|^2 = (-2+2t)^2 + (-7+3t)^2 + (4-6t)^2$$

$69 = 49t^2 - 98t + 69$

$49t(t-2) = 0$

$t = 0$ or $t = 2$

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ or $\overrightarrow{OP} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix}$ **(ans)**

27.

(a) $\mathbf{a} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}$

$\mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Line AB: $\mathbf{r} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}, \ \lambda \in \Re$

$$\overrightarrow{OC} = \begin{pmatrix} 9-3\lambda \\ -2+4\lambda \\ 1+5\lambda \end{pmatrix}$$

$$\begin{pmatrix} 9-3\lambda \\ -2+4\lambda \\ 1+5\lambda \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = 0$$

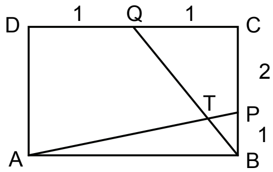
$-27 + 9\lambda - 8 + 16\lambda + 5 + 25\lambda = 0$

$50\lambda = 30$

$\lambda = \frac{3}{5}$

$\overrightarrow{OC} = \begin{pmatrix} 9-\frac{9}{5} \\ -2+\frac{12}{5} \\ 1+3 \end{pmatrix} = \begin{pmatrix} \frac{36}{5} \\ \frac{2}{5} \\ 4 \end{pmatrix}$ **(ans)**

(b)



$AB = 2a$

Let $AT : TP = 1-\lambda : \lambda$, and $BT : TQ = 1-\mu : \mu$

$\overrightarrow{AT} = (1-\lambda)\overrightarrow{AP} = (1-\lambda)\begin{pmatrix} 2a \\ \frac{2a}{3} \end{pmatrix}$

$= (1-\mu)\overrightarrow{AQ} + \mu\overrightarrow{AB}$

$= (1-\mu)\begin{pmatrix} a \\ 2a \end{pmatrix} + \mu\begin{pmatrix} 2a \\ 2 \end{pmatrix}$

$2a(1-\lambda) = (1-\mu)a + 2\mu a$

$2(1-\lambda) = 1+\mu$

$2-2\lambda = 1+\mu$

$\mu = 1-2\lambda$ – Equation 1

$\frac{2}{3}a(1-\lambda) = (1-\mu)(2a)$

$1-\lambda = 3-3\mu$

$3\mu = 2+\lambda$ – Equation 2

Solving Equations 1 and 2,

$1-2\lambda = \frac{1}{3}(2+\lambda)$

$3-6\lambda = 2+\lambda$

$7\lambda = 1$

$\lambda = \frac{1}{7}$

$AT : TP = \frac{6}{7} : \frac{1}{7} = 6:1$ **(ans)**

28.

(a) $\cos\theta = \dfrac{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{5}\sqrt{6}} = \dfrac{-3}{\sqrt{30}}$

Acute angle required

$= 180 - 123.2 = 56.8°$ **(ans)**

(b) When $\lambda = -1$,

$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

Therefore, l_1 passes through the point P.

Let M be any point on l_2.

$\overrightarrow{OM} = \begin{pmatrix} 1+\mu \\ -2\mu \\ 4+\mu \end{pmatrix}$

$\overrightarrow{PM} = \begin{pmatrix} 1+\mu \\ -2\mu \\ 4+\mu \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} \mu \\ 4-2\mu \\ 2+\mu \end{pmatrix}$

$\left|\overrightarrow{PM}\right| = \sqrt{\mu^2 + (4-2\mu)^2 + (2+\mu)^2}$

$= \sqrt{6\mu^2 - 12\mu + 20}$ (shown)

Since $\sqrt{6\mu^2 - 12\mu + 20} = \sqrt{6(\mu-1)^2 + 14}$,

Shortest distance $= \sqrt{14}$ when $\mu = 1$

Alternatively, by using differentiation,

$\dfrac{d}{dx}\sqrt{6\mu^2 - 12\mu + 20} = \dfrac{12\mu - 12}{2\sqrt{6\mu^2 - 12\mu + 20}}$

For the shortest distance,

$\dfrac{d}{dx}\sqrt{6\mu^2 - 12\mu + 20} = 0$ and

$\dfrac{d^2}{dx^2}\sqrt{6\mu^2 - 12\mu + 20} > 0$, therefore the shortest distance occurs when $\dfrac{12\mu - 12}{2\sqrt{6\mu^2 - 12\mu + 20}} = 0$, that is, when $\mu = 1$. Hence, the shortest distance is

$\sqrt{6(1)^2 - 12(1) + 20} = \sqrt{14}$ **(ans)**

29.

(a) $l_1 : \mathbf{r} = \begin{pmatrix} -8 \\ 5 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\lambda \in \Re$

z-axis: $\mathbf{r} = \beta\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\cos\theta = \left|\dfrac{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{9}\sqrt{1}}\right| = \frac{1}{3}$

$\theta = 70.5°$ **(ans)**

(b) Let N be the foot of the perpendicular of the origin from l. Since N lies on l,

$$\overrightarrow{ON}=\begin{pmatrix}-8+2\lambda\\5-2\lambda\\-1+\lambda\end{pmatrix}\text{ for some }\lambda$$

$$\begin{pmatrix}-8+2\lambda\\5-2\lambda\\-1+\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\-2\\1\end{pmatrix}=0$$

$-16+4\lambda-10+4\lambda-1+\lambda=0$

$9\lambda=27$

$\lambda=3$

$$\overrightarrow{ON}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$$ **(ans)**

(c) Let $\begin{pmatrix}-8+2\lambda\\5-2\lambda\\-1+\lambda\end{pmatrix}=\begin{pmatrix}6\alpha\\-4\alpha\\\alpha\end{pmatrix}$.

$-8+2\lambda=6\alpha$ – Equation 1

$5-2\lambda=-4\alpha$ – Equation 2

$-1+\lambda=\alpha$ – Equation 3

Equation 1 + Equation 2:

$-3=2\alpha$

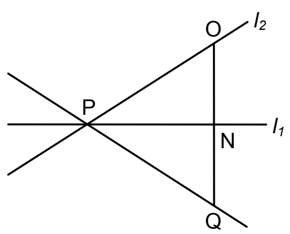
$\alpha=-\frac{3}{2}$

$\lambda=-\frac{1}{2}$

Substituting into Equation 3,

$-1+\lambda=\alpha=-\frac{3}{2}$

Point of intersection: $\overrightarrow{OP}=\begin{pmatrix}-9\\6\\-\frac{3}{2}\end{pmatrix}$



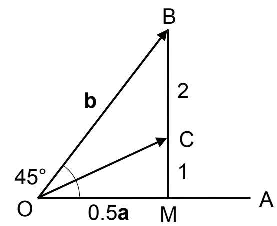
$$\overrightarrow{OQ}=2\overrightarrow{ON}=\begin{pmatrix}-4\\-2\\4\end{pmatrix}$$

$$\overrightarrow{QP}=\begin{pmatrix}-9\\6\\-\frac{3}{2}\end{pmatrix}-\begin{pmatrix}-4\\-2\\4\end{pmatrix}=\begin{pmatrix}-5\\8\\-\frac{11}{2}\end{pmatrix}$$

Equation of mirror line:

$$\mathbf{r}=\begin{pmatrix}-9\\6\\-\frac{3}{2}\end{pmatrix}+\gamma\begin{pmatrix}-10\\16\\-11\end{pmatrix},\ \gamma\in\Re$$ **(ans)**

30.



$\overrightarrow{OC}=\frac{1}{3}\mathbf{b}+\frac{2}{3}\left(\frac{1}{2}\mathbf{a}\right)=\frac{1}{3}(\mathbf{a}+\mathbf{b})$

$-\mathbf{b}\bullet\mathbf{c}=-\mathbf{b}\bullet\frac{1}{3}(\mathbf{a}+\mathbf{b})$

$=-\frac{1}{3}[\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}]$

$=-\frac{1}{3}\left[|\mathbf{b}|\bullet|\mathbf{a}|\cos45^\circ+|\mathbf{b}|^2\right]$

$=-\frac{1}{3}\left[(5)(6)\frac{\sqrt{2}}{2}+5^2\right]$

$=-\frac{1}{3}\left[15\sqrt{2}+25\right]$

$=-\frac{5}{3}\left(3\sqrt{2}+5\right)$ **(ans)**

31.

(a) $l_1:\mathbf{r}=\begin{pmatrix}3\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}-2\\4\\-1\end{pmatrix},\ \lambda\in\Re$

$l_2:\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\mu\begin{pmatrix}2\\1\\1\end{pmatrix},\ \mu\in\Re$

Suppose l_1 and l_2 intersect,

$$\begin{pmatrix}3-2\lambda\\4\lambda\\-2-\lambda\end{pmatrix}=\begin{pmatrix}1+2\mu\\\mu\\-3+\mu\end{pmatrix}$$

$3-2\lambda=1+2\mu$ – Equation 1

$4\lambda=\mu$ – Equation 2

$-2-\lambda=-3+\mu$ – Equation 3

Solving Equations 1 and 2,

$\lambda=\frac{1}{5}$

$\mu=\frac{4}{5}$

Substituting into Equation 3,

$LHS = -2 - \frac{1}{5} = -\frac{11}{5}$

$RHS = -3 + \frac{4}{5} = -\frac{11}{5}$

Since $LHS = RHS$, l_1 and l_2 intersect.

Coordinates of point of intersection:

$C\left(\frac{13}{5}, \frac{4}{5}, -\frac{11}{5}\right)$ **(ans)**

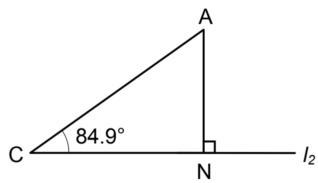
(b) Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{\left|\begin{pmatrix}-2\\4\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\sqrt{(-2)^2+4^2+(-1)^2}\sqrt{2^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{21}\sqrt{6}}$$

$\theta = 84.9°$ (to the nearest $0.1°$)

Let N be the foot of the perpendicular from A to l_2.



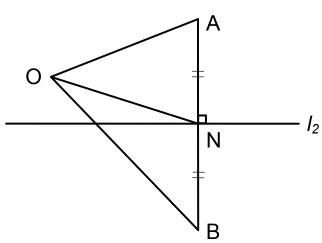
$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \begin{pmatrix}3\\0\\-2\end{pmatrix} - \begin{pmatrix}\frac{13}{5}\\\frac{4}{5}\\-\frac{11}{5}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\-4\\1\end{pmatrix}$$

$$\left|\overrightarrow{CA}\right| = \frac{1}{5}\sqrt{2^2+(-4)^2+1^2} = \frac{\sqrt{21}}{5}$$

$$\sin\angle ACN = \frac{AN}{AC}$$

$$AN = \frac{\sqrt{21}}{5}\sin 84.9° = 0.913 \quad \textbf{(ans)}$$

(c) Let B be the reflection of A in the line l_2.



Since N lies on l_2,

Let $\overrightarrow{ON} = \begin{pmatrix}1+2\mu\\\mu\\-3+\mu\end{pmatrix}$ for some $\mu \in \Re$.

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix}1+2\mu\\\mu\\-3+\mu\end{pmatrix} - \begin{pmatrix}3\\0\\-2\end{pmatrix} = \begin{pmatrix}-2+2\mu\\\mu\\-1+\mu\end{pmatrix}$$

Since $\overrightarrow{AN} \perp l_2$,

$$\overrightarrow{AN}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}-2+2\mu\\\mu\\-1+\mu\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix} = 0$$

$-4+4\mu+\mu-1+\mu = 0$

$\mu = \frac{5}{6}$

$$\overrightarrow{ON} = \begin{pmatrix}\frac{8}{3}\\\frac{5}{6}\\-\frac{13}{6}\end{pmatrix}$$

By Ratio Theorem,

$$\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$

$$\overrightarrow{OB} = 2\overrightarrow{ON} - \overrightarrow{OA} = 2\begin{pmatrix}\frac{8}{3}\\\frac{5}{6}\\-\frac{13}{6}\end{pmatrix} - \begin{pmatrix}3\\0\\-2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}7\\5\\-7\end{pmatrix}$$

$\overrightarrow{OB} = \frac{1}{3}(7\mathbf{i}+5\mathbf{j}-7\mathbf{k})$ **(ans)**

32.

(a) $l_1 : \mathbf{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\lambda \in \Re$

$\overrightarrow{ON} = \begin{pmatrix}1+\lambda\\3+2\lambda\\4-\lambda\end{pmatrix}$ for some $\lambda \in \Re$

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix}2+\lambda\\2+2\lambda\\-6-\lambda\end{pmatrix}$$

$$\overrightarrow{AN}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix} = 0$$

$2+\lambda+4+4\lambda+6+\lambda = 0$

$\lambda = -2$

$\overrightarrow{ON} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$ **(ans)**

(b) $\overrightarrow{OB}=\begin{pmatrix}1\\3\\4\end{pmatrix}$, $\overrightarrow{PB}=\begin{pmatrix}-4\\-8\\4\end{pmatrix}$

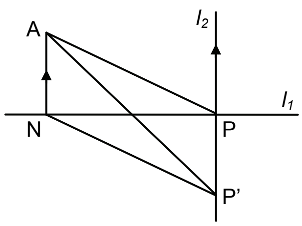
$\overrightarrow{AB}=\begin{pmatrix}2\\2\\-6\end{pmatrix}$

$\cos\angle PBA=\dfrac{\overrightarrow{PB}\bullet\overrightarrow{AB}}{\left|\overrightarrow{PB}\right|\left|\overrightarrow{AB}\right|}=\dfrac{-8-16-24}{\sqrt{96}\sqrt{44}}$

$=-0.73854$

$\angle PBA=137.6^\circ$ **(ans)**

(c)



Area of $\Delta APN=$ Area of $\Delta AP'N$

l_2 is parallel to $\overrightarrow{AN}$.

$\overrightarrow{AN}=\begin{pmatrix}0\\-2\\-4\end{pmatrix}=-2\begin{pmatrix}0\\1\\2\end{pmatrix}$

$l_2:\mathbf{r}=\begin{pmatrix}5\\11\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}$, $\mu\in\Re$ **(ans)**

33.

(a) $\dfrac{x-8}{2}=4b-y=z-3$

$x=8+2\mu$

$y=4b-\mu$

$z=3+\mu$

Equation of l_2:

$\mathbf{r}=\begin{pmatrix}8\\4b\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\mu\in\Re$

Since l_2 passes through $B(6,-3,2)$,

$\begin{pmatrix}6\\-3\\2\end{pmatrix}=\begin{pmatrix}8\\4b\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$2=3+\mu$

$\mu=-1$

$-3=4b+(-1)(-1)$

$4b=-4$

$b=-1$

Alternatively,

Substituting $x=6$, $y=-3$ and $z=2$ into equation of l_2,

$\dfrac{6-8}{2}=4b-(-3)=2-3$

$-1=4b+3=-1$

$b=-1$

Equation of l_2:

$\mathbf{r}=\begin{pmatrix}8\\-4\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\mu\in\Re$ **(ans)**

(b) Let $\begin{pmatrix}4\\-2\\1\end{pmatrix}=\begin{pmatrix}9\\-2\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$

Unique answer:

$\lambda=-5$

$A(4,-2,1)$ lies on l_1.

Let $\begin{pmatrix}4\\-2\\1\end{pmatrix}=\begin{pmatrix}8\\-4\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$

Unique answer:

$\mu=-2$

$A(4,-2,1)$ lies on l_2.

Since $A(4,-2,1)$ lies on both l_1 and l_2, l_1 and l_2 intersect at $A(4,-2,1)$. **(ans)**

(c) Since C is on l_1, $\overrightarrow{OC}=\begin{pmatrix}9+\lambda\\-2\\6+\lambda\end{pmatrix}$.

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}9+\lambda\\-2\\6+\lambda\end{pmatrix}-\begin{pmatrix}6\\-3\\2\end{pmatrix}=\begin{pmatrix}3+\lambda\\1\\4+\lambda\end{pmatrix}$

$\overrightarrow{BC}$ is parallel to l_2.

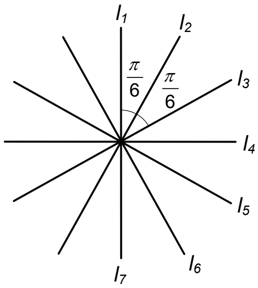
$\begin{pmatrix}3+\lambda\\1\\4+\lambda\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$

$6+2\lambda-1+4+\lambda=0$

$\lambda=-3$

$\overrightarrow{OC}=\begin{pmatrix}6\\-2\\3\end{pmatrix}$ **(ans)**

(d) Let θ be the angle between l_1 and l_2.



$$\cos\theta = \frac{\begin{pmatrix}1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}}{\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}$$

Least value of $n = 7$ **(ans)**

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34. $l_1 : \mathbf{r} = \begin{pmatrix}2\\2\\p\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\q\end{pmatrix}$

$\prod_1 : \mathbf{r}\bullet\begin{pmatrix}2\\2\\p\end{pmatrix} = 2$

Since l_1 lies on $\prod_1$,

$\begin{pmatrix}2\\2\\p\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix} = 2$

$p = -2$ (shown)

Also, since $l_1 \perp \begin{pmatrix}1\\1\\1\end{pmatrix}$,

$\begin{pmatrix}1\\0\\q\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix} = 0$

$q = -1$

$l_1 : \mathbf{r} = \begin{pmatrix}2\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$

When $\lambda = -3$, $\mathbf{r} = \begin{pmatrix}-1\\2\\1\end{pmatrix} = \overrightarrow{OA}$ (verified)

Normal of $\prod_2$:

$n_2 = \begin{pmatrix}2\\2\\-2\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-2\\0\\-2\end{pmatrix} = -2\begin{pmatrix}1\\0\\1\end{pmatrix}$

Thus, $\prod_2 : \mathbf{r}\bullet\begin{pmatrix}1\\0\\1\end{pmatrix} = 0$

And, $\prod_3 : \mathbf{r}\bullet\begin{pmatrix}1\\5\\1\end{pmatrix} = \begin{pmatrix}2\\1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\5\\1\end{pmatrix} = 10$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-1\\3\\2\end{pmatrix} - \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$

Length of projection of $\overrightarrow{AB}$ on $\prod_3$

$$= \left|\overrightarrow{AB}\times\frac{n_3}{|n_3|}\right| = \left|\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\5\\1\end{pmatrix}\frac{1}{\sqrt{27}}\right|$$

$$= \frac{1}{\sqrt{27}}\left|\begin{pmatrix}-4\\1\\-1\end{pmatrix}\right| = \sqrt{\tfrac{18}{27}} = \sqrt{\tfrac{2}{3}}$$

$x + y + z = 2$, $x + z = 0$ and $x + 5y + z = 10$ intersect at l_1.

Alternatively,

For $\prod_3$,

$\begin{pmatrix}2\\2\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\5\\1\end{pmatrix} = 10$

$\begin{pmatrix}2\\2\\-2\end{pmatrix}$ lies on $\prod_3$.

When $\lambda = 1$,

$\begin{pmatrix}3\\2\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\5\\1\end{pmatrix} = 10$

$\begin{pmatrix}3\\2\\-3\end{pmatrix}$ also lies on $\prod_3$.

Since $\begin{pmatrix}2\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\2\\-3\end{pmatrix}$ lies both on l_1 and $\prod_3$, $\prod_1$, $\prod_2$ and $\prod_3$ all intersect on the line l_1.

Solving $\begin{pmatrix}1 & 1 & 1\\1 & 0 & 1\\1 & 5 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\0\\10\end{pmatrix}$,

$y=2$, $x+z=0$

Let $\mu=z$, $x=-\mu$

Thus, $\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}$ which is line l_1.

$\prod_1$, $\prod_2$ and $\prod_3$ all intersect on the line l_1. **(ans)**

35.

(a) Substitute the equation of line into that of plane:

$$\left[\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}\right]\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=3 \text{ and get } \lambda=3-2\mu.$$

Substitute $\lambda=3-2\mu$ into

$$\prod_2: \mathbf{r}=(3-2\mu)\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

Alternatively,

Substituting $\mu=\frac{1}{2}(3-\lambda)$ into $\prod_2$,

$$\mathbf{r}=\tfrac{3}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

Shortest distance

$$=\left|\overrightarrow{AB}\times\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|=\left|\begin{pmatrix}0\\2\\6\end{pmatrix}\times\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}-2\\0\\-4\end{pmatrix}\right|=\sqrt{\tfrac{10}{3}} \quad \textbf{(ans)}$$

(b) Let $\overrightarrow{OC}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\-1\end{pmatrix}$ for some $\alpha\in\Re$.

$\overrightarrow{BA}\bullet\overrightarrow{BC}=0$

$$\begin{pmatrix}0\\-2\\0\end{pmatrix}\bullet\left[\begin{pmatrix}0\\-2\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\-1\end{pmatrix}\right]=0$$

Solving,

$$\alpha=2 \text{ and } \overrightarrow{OC}=\begin{pmatrix}7\\3\\-2\end{pmatrix}$$

Since $\overrightarrow{CD}=\overrightarrow{AB}$,

$$\overrightarrow{OD}-\begin{pmatrix}7\\3\\-2\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\overrightarrow{OD}=\begin{pmatrix}7\\5\\-2\end{pmatrix} \quad \textbf{(ans)}$$

(c) Observe that O lies on $\prod_2$. The midpoint of OA which is $\left(\frac{3}{2},\frac{1}{2},0\right)$ then lies on $\prod_3$.

Compute $n_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

$$\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\\frac{1}{2}\\0\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=1 \text{ (shown)}$$

Let F be the foot of the perpendicular from A to $\prod_3$ and pick a point $E(1,0,0)$ on $\prod_3$.

$$\overrightarrow{AF}=\left[\overrightarrow{AE}\left(\frac{1}{\sqrt{3}}\right)\begin{pmatrix}1\\-1\\1\end{pmatrix}\right]\left(\frac{1}{\sqrt{3}}\right)\begin{pmatrix}1\\-1\\1\end{pmatrix}=-\tfrac{1}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Thus, $\overrightarrow{OF}=\overrightarrow{OA}+\overrightarrow{AF}=\frac{1}{3}\begin{pmatrix}8\\4\\-1\end{pmatrix}$ **(ans)**

36.

(a) Direction vector of l_1 : $\begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$

Equation of $l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}, \ \lambda \in \Re$ **(ans)**

(b) $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 13 \\ -3 \end{pmatrix}$

$\overrightarrow{ON} = \begin{pmatrix} 2 \\ -1-2\lambda \\ 3+5\lambda \end{pmatrix}$

$\overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB} = \begin{pmatrix} -2 \\ -14-2\lambda \\ 6+5\lambda \end{pmatrix}$

$\overrightarrow{BN} \bullet \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} = 0$

$\begin{pmatrix} -2 \\ -14-2\lambda \\ 6+5\lambda \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} = 0$

$28 + 4\lambda + 30 + 25\lambda = 0$

$29\lambda = -58$

$\lambda = -2$

$\overrightarrow{BN} = \begin{pmatrix} -2 \\ -10 \\ -4 \end{pmatrix}$

Equation of line BN:

$\mathbf{r} = \begin{pmatrix} 4 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ **(ans)**

37.

(a) $\prod_1 : \mathbf{r} \bullet \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} = 15$

Distance of A from $\prod_1$

$= \left| \frac{\begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}}{\sqrt{50}} - \frac{15}{\sqrt{50}} \right|$

$= \left| \frac{4}{\sqrt{50}} - \frac{15}{\sqrt{50}} \right| = \frac{11}{\sqrt{50}}$

$\frac{15}{\sqrt{50}} > \frac{4}{\sqrt{50}}$

A and O are on the same side of $\prod_1$ **(ans)**

(b) Vector parallel to $\prod_2 = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

Normal vector of $\prod_2$

$= \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ -12 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}$

Angle between $\prod_1$ and $\prod_2$

$= \cos^{-1} \left| \frac{\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}}{\sqrt{50}\sqrt{41}} \right| = \cos^{-1} \left| \frac{31}{\sqrt{(50)(41)}} \right|$

$= 46.8°$ **(ans)**

(c) Equation of $\prod_2$:

$\mathbf{r} \bullet \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}$

$\mathbf{r} \bullet \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix} = 6$

$5x - 4y + 3z = 15$ – Equation 1

$2x - 6y - z = 6$ – Equation 2

$x + 8y + az = b$ – Equation 3

For line of intersection of $\prod_1$ and $\prod_2$:

$$A=\begin{bmatrix}5 & -4 & 3 & 15\\ 2 & -6 & -1 & 6\end{bmatrix}$$

After RREF,

$$A=\begin{bmatrix}1 & 0 & 1 & 3\\ 0 & 1 & \frac{1}{2} & 0\end{bmatrix}$$

$x+z=3$

$x=3-z$

$y+\frac{1}{2}z=0$

$y=-\frac{1}{2}z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3-z\\-\frac{1}{2}z\\z\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

Let $\mu=0,1$.

Two points on common line:

$(3,0,0)$ and $(1,-1,2)$

Substituting into $x+8y+az=b$,

$3+8(0)+0a=b$

$b=3$

$1-8+2a=b$

$2a-b=7$

$2a=10$

$a=5$

Alternatively,

Since the common line lies in $\prod_3$,

$$\left[\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}\right]\begin{pmatrix}1\\8\\a\end{pmatrix}=b \text{ for all } \mu.$$

$3-2\mu-8\mu+2\mu a=b$

$3-\mu(10-2a)=b$

$b=3$ and $a=5$ **(ans)**

38.

(a) Given: $\overrightarrow{OA}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}3\\1\\-1\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}0\\-5\\-7\end{pmatrix}$,

$$\overrightarrow{AB}=\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}-1\\-5\\-5\end{pmatrix}$$

Normal to $\prod_1$:

$$n_1=\begin{pmatrix}2\\1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-5\\-5\end{pmatrix}=\begin{pmatrix}0\\9\\-9\end{pmatrix}=9\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Vector equation of $\prod_1$:

$$\mathbf{r}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}=2 \quad \textbf{(ans)}$$

(b) Equation of line l_1:

$$\mathbf{r}=\begin{pmatrix}1\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$\overrightarrow{OR}=\begin{pmatrix}1+2\lambda\\\lambda\\-2+\lambda\end{pmatrix}$$

$$\overrightarrow{CR}=\begin{pmatrix}1+2\lambda\\\lambda+5\\5+\lambda\end{pmatrix}$$

$\overrightarrow{CR}\perp\overrightarrow{AB}$

$$\begin{pmatrix}1+2\lambda\\\lambda+5\\5+\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix}=0$$

$6\lambda+12=0$

$\lambda=-2$

Hence, $\overrightarrow{OR}=\begin{pmatrix}-3\\-2\\-4\end{pmatrix}$ **(ans)**

(c) Normal to $\prod_2$:

$$n_2 = \overrightarrow{AB} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

Vector equation of $\prod_2$:

$$r \bullet \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\-5\\-7\end{pmatrix} \bullet \begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$r \bullet \begin{pmatrix}2\\1\\1\end{pmatrix} = -12 \quad \textbf{(ans)}$$

(d) $3\overrightarrow{CR} = 2\overrightarrow{CQ}$

$$\overrightarrow{CQ} = \tfrac{3}{2}\overrightarrow{CR}$$

$$\overrightarrow{CQ} = \tfrac{3}{2}\left[\begin{pmatrix}-3\\-2\\-4\end{pmatrix} - \begin{pmatrix}0\\-5\\-7\end{pmatrix}\right] = \tfrac{9}{2}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\overrightarrow{BR} = \begin{pmatrix}-3\\-2\\-4\end{pmatrix} - \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}-6\\-3\\-3\end{pmatrix} = -3\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$CQ = \tfrac{9}{2}\sqrt{3}$$

$$BR = 3\sqrt{6}$$

Area of $\Delta BCQ = \tfrac{1}{2} CQ \times BR$

$$= \tfrac{1}{2}\left(\tfrac{9}{2}\sqrt{3}\right)\left(3\sqrt{6}\right) = \tfrac{81}{4}\sqrt{2}$$

Alternatively,

$$\overrightarrow{CB} = 3\begin{pmatrix}1\\2\\2\end{pmatrix}$$

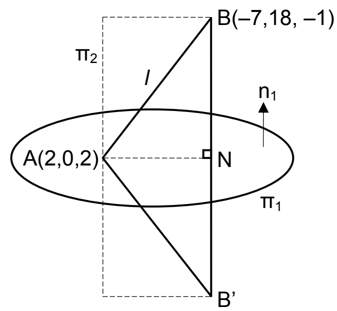
$$\text{Area } = \tfrac{1}{2}\left|\overrightarrow{CB} \times \overrightarrow{CQ}\right| = \tfrac{1}{2}\left|3\begin{pmatrix}1\\2\\2\end{pmatrix} \times \tfrac{9}{2}\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|$$

$$= \tfrac{1}{2}\left(\tfrac{27}{2}\right)\left|\begin{pmatrix}0\\-3\\3\end{pmatrix}\right| = \tfrac{27}{4}\left|\begin{pmatrix}0\\-3\\3\end{pmatrix}\right|$$

$$= \tfrac{27}{4}\sqrt{18} = \tfrac{81}{4}\sqrt{2} \quad \textbf{(ans)}$$

39.

(a)



$$\prod_1 : \mathbf{r} \bullet \begin{pmatrix}1\\-2\\1\end{pmatrix} = 4$$

$$l_{BN} : \mathbf{r} = \begin{pmatrix}-7\\18\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}, \ \lambda \in \Re$$

$$\overrightarrow{ON} = \begin{pmatrix}-7+\lambda\\18-2\lambda\\-1+\lambda\end{pmatrix} \text{ for some } \lambda \in \Re$$

Thus, $\begin{pmatrix}-7+\lambda\\18-2\lambda\\-1+\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\-2\\1\end{pmatrix} = 4$

$$-7+\lambda-2(18-2\lambda)-1+\lambda = 4$$

$$\lambda = 8$$

$$\overrightarrow{ON} = \begin{pmatrix}-7+8\\18-2(8)\\-1+8\end{pmatrix} = \begin{pmatrix}1\\2\\7\end{pmatrix} \text{ (shown)}$$

Perpendicular distance

$$= \left|\overrightarrow{BN}\right| = \left|\overrightarrow{ON} - \overrightarrow{OB}\right|$$

$$= \left|\begin{pmatrix}1\\2\\7\end{pmatrix} - \begin{pmatrix}-7\\18\\-1\end{pmatrix}\right| = \left|8\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|$$

$$= 8\sqrt{1+4+1} = 8\sqrt{6} \quad \textbf{(ans)}$$

(b) $\begin{pmatrix}2\\0\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=2+2=4$

Therefore, point A lies in $\prod_1$. (verified)

$\overrightarrow{AB'}=\overrightarrow{AB}+2\overrightarrow{BN}$

$$=\begin{pmatrix}-7\\18\\-1\end{pmatrix}-\begin{pmatrix}2\\0\\2\end{pmatrix}+2\begin{pmatrix}8\\-16\\8\end{pmatrix}=\begin{pmatrix}7\\-14\\13\end{pmatrix}$$

$\begin{pmatrix}7\\-14\\13\end{pmatrix}$ is parallel to $\begin{pmatrix}7\\14\\-13\end{pmatrix}$ (shown) **(ans)**

(c) $\overrightarrow{AB}=\begin{pmatrix}-7\\18\\-1\end{pmatrix}-\begin{pmatrix}2\\0\\2\end{pmatrix}=\begin{pmatrix}-9\\18\\-3\end{pmatrix}=-3\begin{pmatrix}3\\-6\\1\end{pmatrix}$

Two vectors parallel to $\prod_2$:

$\begin{pmatrix}3\\-6\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$$\begin{pmatrix}3\\-6\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}-4\\-2\\0\end{pmatrix}=-2\begin{pmatrix}2\\1\\0\end{pmatrix}$$

Vector perpendicular to $\prod_2$:

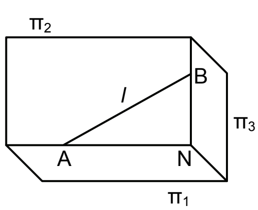
$$n_2=\begin{pmatrix}2\\1\\0\end{pmatrix}$$

Also, point A lies in $\prod_2$.

Therefore, $\mathbf{r}=\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}\bullet\begin{pmatrix}2\\1\\0\end{pmatrix}=4$

Equation of $\prod_2$: $\mathbf{r}\bullet\begin{pmatrix}2\\1\\0\end{pmatrix}=4$ **(ans)**

(d)



Distance A from $\prod_3$

$$=\left|\overrightarrow{AN}\right|=\left|\overrightarrow{ON}-\overrightarrow{OA}\right|$$

$$=\left|\begin{pmatrix}1\\2\\7\end{pmatrix}-\begin{pmatrix}2\\0\\2\end{pmatrix}\right|=\left|\begin{pmatrix}-1\\2\\5\end{pmatrix}\right|$$

$=\sqrt{1+4+25}=\sqrt{30}$ **(ans)**

(e) Acute angle between l_1 and $\prod_3$

$$=\angle ABN=\tan^{-1}\frac{AN}{BN}=\tan^{-1}\frac{\sqrt{30}}{8\sqrt{6}}$$

$=15.6^\circ$ (to 1 decimal place) **(ans)**

40.

(a) $l_1:\mathbf{r}=\begin{pmatrix}p\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\ \lambda\in\Re$

Given: $q=1$ and $p=4$,

Since C lies on l_1,

$$\overrightarrow{OC}=\begin{pmatrix}4\\3\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}4\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$\overrightarrow{AC}\perp l_2$:

$$\begin{pmatrix}4+\lambda\\\lambda\\4\end{pmatrix}\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=0$$

$\lambda=-12$

$\overrightarrow{OC}=-8\mathbf{i}-9\mathbf{j}+7\mathbf{k}$ **(ans)**

(b) $\overrightarrow{AB}=q\mathbf{i}+2\mathbf{k}$

Given:

Acute angle between l_1 and $l_2=60^\circ$

$$\cos 60^\circ=\left|\frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\bullet\begin{pmatrix}q\\0\\2\end{pmatrix}}{\sqrt{2}\sqrt{q^2+4}}\right|$$

$\sqrt{2}\sqrt{q^2+4}=2q$

$q=\pm 2$ **(ans)**
