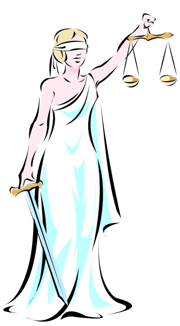
2010

advanced mathematics

teacher's reference
9740

Learning objectives implemented



Learning objectives implemented in 2010 Part 1 9740/1

Questions

Answer <u>all</u> questions.

1. [Vectors]

Method

Approach I – vector products

In a vector space, it is given that $|\mathbf{a}| = |\mathbf{b}|$ and the position vectors **a** and **b** are given by

$\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k}$ and

$\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

where $p > 0$.

(i) $\therefore$ To find the exact value of p:

The magnitude, $|\mathbf{a}| = \sqrt{(2p)^2 + (3p)^2 + (6p)^2}$

$= \pm p\sqrt{4+9+36} = \pm p\sqrt{49} = \pm 7p$

The magnitude, $|\mathbf{b}| = \sqrt{(1)^2 + (-2)^2 + (2)^2}$

$= \sqrt{1+4+4} = 3 \text{ units} \equiv |\mathbf{a}| = \pm 7p$ — ❶

$\therefore$ **The exact value of p** (from ❶)

$= \left|\frac{3}{\pm 7}\right|$ $(p > 0)$

$= \frac{3}{7}$ unit **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

Some candidates suggested the answer as $\pm\frac{3}{7}$.

The answer mark was not credited.

☺ **Checking the Answer**

- Wherever possible do check the answer by substituting a simple numerical solution to the original equation and test its validity.

(ii) $\therefore$ **To show that $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) = 0$:** — ❶

LHS of ❶ $= (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})$

$= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{b}$

$= |\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 = |\mathbf{a}|^2 - |\mathbf{b}|^2$

$= 0 =$ RHS of ❶ **(QED)** [3]

☺ **Common misconception**

Q.E.D. does not mean "*quite easily done*", but is an acronym of the Latin phrase *quod erat demonstrandum*, which means "that which was to be demonstrated". The phrase is traditionally placed in its abbreviated form at the end of a mathematical proof or philosophical argument when that which was specified in the enunciation, and in the setting–out, has been exactly restated as the conclusion of the demonstration. The abbreviation thus signals the completion of the proof.

§

2. [Sequences & Series]

Method

Approach I – Maclaurin Series

(i) [For this question part, you may use the standard results given in the List of Formulae (MF15).]

Given: $e^x(1+\sin 2x)$ — ❶

$\therefore$ To find the first three terms of the Maclaurin series:

$e^x(1+\sin 2x)$

$$= e^x\left\{1 + \left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots\right]\right\}$$

$$= \left\{1 + x + \left[\frac{x^2}{2!}\right] + \ldots\right\}\left\{1 + \left[2x - \tfrac{8}{6}x^3 + \tfrac{32}{120}x^5 + \ldots\right]\right\}$$

$$= \left\{1 + x + \tfrac{1}{2}x^2 + \ldots\right\}\left\{1 + 2x - \tfrac{4}{3}x^3 + \tfrac{4}{15}x^5 + \ldots\right\}$$

$$= \left[1 + 2x - \tfrac{4}{3}x^3\right] + x\left[1 + 2x - \tfrac{4}{3}x^3\right] + \tfrac{1}{2}x^2\left[1 + 2x - \tfrac{4}{3}x^3\right] + \ldots$$

$$= 1 + 2x - \tfrac{4}{3}x^3 + x + 2x^2 - \tfrac{4}{3}x^4 + \ldots + \tfrac{1}{2}x^2 + x^3 + \ldots$$

$$= 1 + 3x + \tfrac{5}{2}x^2 + \ldots$$

$\therefore$ **The first three terms of the Maclaurin series**

$\approx 1 + 3x + \frac{5}{2}x^2$ (correct to 3 terms) **(ans)** [3]

☺ **CheckBack**

If the answer is $1 + 3x + \frac{5}{2}x^2$,

Choose a very small radian measure, such as 0.1 rad.

❶: $e^x(1 + \sin 2x) = e^{0.1}(1 + \sin 2(0.1))$

$= 1.32473$

Applying the value of x to the three terms,

$1 + 3x + \frac{5}{2}x^2$

$= 1 + 3(0.1) + 2.5(0.1)^2$

$= 1.325$ ($\Delta 0.02\%$, close enough) **(checked)**

(ii) Given: $\left(1 + \frac{4}{3}x\right)^n$ — ❷

The first two terms in the series expansion, the ascending powers of x, of ❷ are equal to the first two terms in the series expansion of ❶.

$\therefore$ To find the value of n:

❷: $\left(1+\frac{4}{3}x\right)^n$

$$= \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}\left(\tfrac{4}{3}x\right)^1 + \binom{n}{2}(1)^{n-2}\left(\tfrac{4}{3}x\right)^2 + \ldots$$

$$= 1 + n\left(\tfrac{4}{3}x\right) + \frac{n(n-1)}{2!}\left(\tfrac{4}{3}x\right)^2 + \ldots$$

Comparing coefficient for the second term,

$\therefore$ **The value of n**

$\Rightarrow n\left(\frac{4}{3}\right) = 3 \Rightarrow n = \frac{9}{4}$ **(ans)**

$\therefore$ **To show that the third terms in each of the series are equal:**

$\therefore$ The third term of ❶ $= \frac{5}{2}x^2$

The third term of ❷ $= \dfrac{n(n-1)}{2!}\left(\tfrac{4}{3}x\right)^2$

$$= \frac{\frac{9}{4}\left(\frac{9}{4}-1\right)}{2!} \times \tfrac{16}{9}x^2 = \frac{\frac{9}{4}\left(\frac{5}{4}\right)}{2} \times \tfrac{16}{9}x^2 = \tfrac{5}{2}x^2 \quad \textbf{(QED)}$$

[3]

☺ **Exam Report**

Quite an unusual large number of candidates did not get the correct terms for the question. Most credits were lost due to careless algebra.

§

3. [Sequences & Series]

Method

Approach I – recurrence functions

Given a sequence: $u_1, u_2, u_3, \ldots$ — ❶

The sum, S_n, of the first n terms of a sequence ❶ is given by

$S_n = n(2n + c)$, where c is a constant. — ❷

(i) $\therefore$ To find u_n, in terms of c and n:

❷: $S_n = n(2n + c)$

$S_{n-1} = (n-1)[2(n-1) + c]$

$\therefore$ **u_n, in terms of c and n**

$= S_n - S_{n-1}$

$= n(2n + c) - (n-1)[2(n-1) + c]$

$= 2n^2 + cn - (n-1)[2n - 2 + c]$

$= 2n^2 + cn - [(2n^2 - 2n + cn) - (2n - 2 + c)]$

$= 2n^2 + cn - [2n^2 - 2n + cn - 2n + 2 - c]$

$= 2n^2 + cn - [2n^2 - 4n + cn + 2 - c]$

$= 2n^2 + cn - 2n^2 + 4n - cn - 2 + c$

$= 4n - 2 + c$ **(ans)** [3]

☺ **CheckBack**

If the answer is $4n - 2 + c$,

- Let $n = 100$,

❷: $S_{100} = n(2n + c)$

$= 100[2(100) + c] = 100[200 + c]$

$S_{99} = 99[2(99) + c]$

$= 99[198 + c]$

$u_{100} = S_{100} - S_{99}$

$= 100[200 + c] - 99[198 + c]$

$= 398 + c$

$u_n = 4n - 2 + c$

$= 4(100) - 2 + c = 398 + c$ **(checked)**

☺ **Exam Report**

Most candidates gave the correct answer.

(ii) $\therefore$ To find a recurrence relation of the form $u_{n+1} = f(u_n)$:

$u_n = 4n - 2 + c$

$u_{n+1} = 4(n+1) - 2 + c$

$= 4n + 4 - 2 + c = (4n - 2 + c) + 4$

$= u_n + 4$

$\therefore$ **A recurrence relation of the form $u_{n+1} = f(u_n)$**

$u_{n+1} = u_n + 4$ **(ans)** [2]

☺ **CheckBack**

There is no convenient CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer.

§

4. [Calculus]

Method

Approach I – differentiation

Given that a curve is defined as:

$x^2 - y^2 + 2xy + 4 = 0$ — ❶

(i) $\therefore$ To find the derivative of ❶:

Differentiating ❶ wrt x:

$$2x - 2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} + 0 = 0$$

$$2x + 2y + (2x - 2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2y + 2x)}{2y - 2x}$$

$\therefore$ **The derivative of ❶:**

$\frac{dy}{dx} = \frac{y + x}{y - x}$ **(ans)** [4]

☺ **Exam Report**

Many candidates left the answer as:

$$\frac{dy}{dx} = -\frac{y + x}{x - y}$$

The final answer mark was not credited.

(ii) $\therefore$ Let P be the exact coordinates of the points on the curve where the tangent of the curve is parallel to the x–axis.

For tangents parallel to the x–axis,

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = 0 = \frac{y + x}{y - x} \Rightarrow 0 = y + x \Rightarrow y = -x$$

Substituting the condition, $y = -x$, back into ❶:

$$x^2 - (-x)^2 + 2x(-x) + 4 = 0$$

$$x^2 - x^2 - 2x^2 + 4 = 0$$

$$-2x^2 + 4 = 0$$

$$2x^2 = 4$$

$$x = \pm\sqrt{2} \Rightarrow y = \mp\sqrt{2}$$

$\therefore$ **The exact coordinates of the points on the curve where the tangent of the curve is parallel to the x–axis**

$P \equiv (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ respectively.

(ans) [4]

☺ **CheckBack**

There is no convenient CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer.

§

5. [Functions and Graphs]

Method

Approach I – transformations

The curve with equation $y = x^3$ (⓿) is made to undergo 3 transformations:

❶ transformed by a translation of 2 units in the positive x–direction,

❷ followed by a stretch with scale factor ½ parallel to the y–axis,

❸ followed by a translation of 6 units in the negative y–direction.

(i) ∴ Let E be the equation of the new curve in the form of $y = \mathrm{f}(x)$.

❶ $x \to x - 2 \quad \Rightarrow \quad y = \mathrm{f}(x{-}2) = (x-2)^3$

❷ $y \to ½y \quad \Rightarrow \quad y = ½\mathrm{f}(x') = ½(x-2)^3$

❸ $y \to x - 6 \quad \Rightarrow \quad y = \mathrm{f}(x'') + 6 = ½(x-2)^3 - 6$

∴ **The equation of the new curve in the form of y = f(x)**

E: $y = ½(x-2)^3 - 6$ **(ans)**

∴ Let P be the exact coordinates of the points where the curve crosses the x– and y– axes.

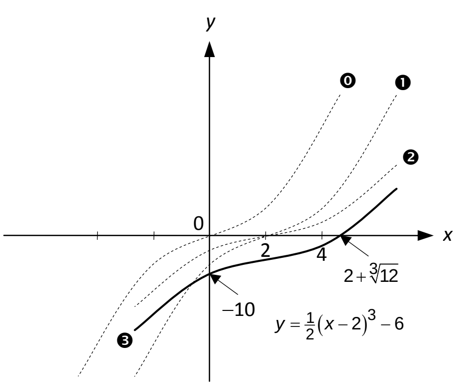
- When $x = 0$, $y = ½(0-2)^3 - 6 = -10$
- When $y = 0$, $0 = ½(x-2)^3 - 6 \quad \Rightarrow \quad x = 2 + \sqrt[3]{12}$

∴ **The exact coordinates of the points where the curve crosses the x– and y– axes**

$P \equiv \left(2 + \sqrt[3]{12}\,, 0\right)$ and $\left(0\,, -10\right)$ respectively.

(ans)

∴ **The sketch of the curve:**



(ans) [5]

☺ **CheckBack**

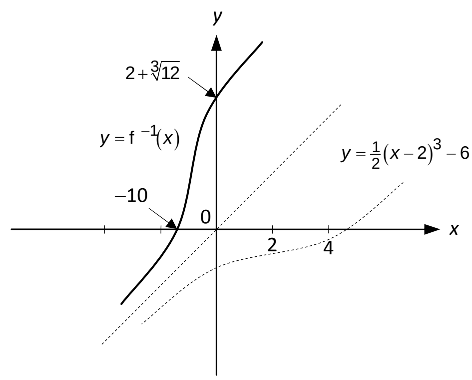
Transform the final curve using the transformations given and deduce the final curve form.

The constants and variables can also be checked.

(ii) On the same diagram,

∴ **The sketch of the graph $\mathrm{f}^{-1}(x)$:**

The graph of $\mathrm{f}^{-1}(x)$ is a reflection about the $y = x$ line.



∴ **The exact coordinates of the points where the graph crosses the x– and y– axes:**

$P' \equiv \left(-10, 0\right)$ and $\left(0, 2 + \sqrt[3]{12}\right)$ respectively.

(ans) [3]

☺ **Exam Report**

Most errors occurred at the deduction of transformation part.

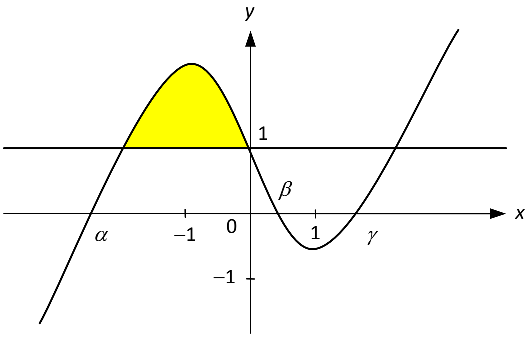
Candidates were advised to comprehend the curve transformations both physically and algebraically to save time.

§

6. [Functions and Graphs]

Method

Approach I – roots



The diagram above shows a curve and a line.

The curve has equation defined as:

$$y = x^3 - 3x + 1 \qquad — ❶$$

The line has equation defined as:

$$y = 1$$

The curve crosses the x–axis at $x = \alpha$, $x = \beta$, and $x = \gamma$, and has turning points at $x = -1$ and $x = 1$.

(i) [For this question, give answers correct to 3 decimal places]

$\therefore$ To deduce the values of β and γ:

Rearranging ❶,

$$y = x^3 - 3x + 1 = 0$$

$$x = \sqrt[3]{3x - 1}$$

Let $\gamma_0 = 10$,

$$\gamma_1 = \sqrt[3]{3\gamma_0 - 1} = \sqrt[3]{3(10) - 1} = 3.107233$$

$$\gamma_2 = \sqrt[3]{3\gamma_1 - 1} = 2.104578$$

$\gamma_3 = 1.848256$, $\gamma_4 = 1.769950$, $\gamma_5 = 1.744593$,
$\gamma_6 = 1.736221$, $\gamma_7 = 1.733440$, $\gamma_8 = 1.732514$,
$\gamma_9 = 1.732205$, $\gamma_{10} = 1.732102$, $\gamma_{11} = 1.732068$

$\gamma \approx 1.732$ (3dp)

Let $\alpha_0 = -10$,

$$\alpha_1 = \sqrt[3]{3\alpha_0 - 1} = \sqrt[3]{3(-10) - 1} = -3.107233$$

$$\alpha_2 = \sqrt[3]{3\alpha_1 - 1} = -2.104578$$

$\alpha_3 = -1.848256$, $\alpha_4 = -1.769950$
$\alpha_5 = -1.744593$, $\alpha_6 = -1.736221$,
$\alpha_7 = -1.733440$, $\alpha_8 = -1.732514$,
$\alpha_9 = -1.732205$, $\alpha_{10} = -1.732102$,
$\alpha \approx -1.732$ (3dp)

Rearranging ❶,

$$y = x^3 - 3x + 1 = 0$$

$$x = \tfrac{1}{3}\left(x^3 + 1\right)$$

Let $\beta_0 = 0$,

$$\beta_1 = \tfrac{1}{3}\left(\beta_0^{\,3} + 1\right) = \tfrac{1}{3}\left(0^3 + 1\right) = 0.333333$$

$$\beta_2 = \tfrac{1}{3}\left(\beta_1^{\,3} + 1\right) = 0.345679$$

$\beta_3 = 0.347102$, $\beta_4 = 0.347273$, $\beta_5 = 0.347294$

$\beta \approx 0.347$ (3dp)

OR

Using the graphic calculator,

$\therefore$ **The values of β and γ:**

0.347 and 1.732 respectively **(ans)** [2]

(ii) [For this question, give answer correct to 3 decimal places]

$\therefore$ Let A be the area of the region bounded by the curve and the x–axis between $x = \beta$ and $x = \gamma$.

Area of the bounded region $= \int_{\beta}^{\gamma} y \, dx$

$$= \int_{0.347}^{1.732} x^3 - 3x + 1 \; dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{3}{2}x^2 + x\right]_{0.347}^{1.732}$$

$$= \left[\tfrac{1}{4}(1.732)^4 - \tfrac{3}{2}(1.732)^2 + (1.732)\right] - \left[\tfrac{1}{4}(0.347)^4 - \tfrac{3}{2}(0.347)^2 + (0.347)\right]$$

$$= -0.51800 - 0.17001$$

$$= -0.68801$$

OR

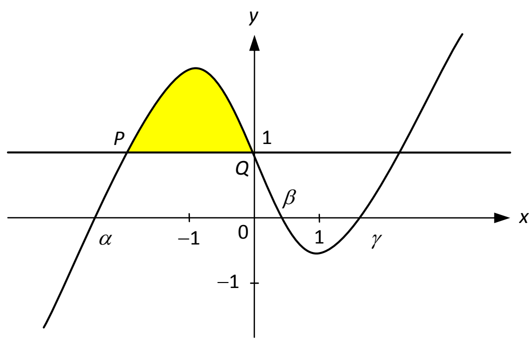
Using the graphic calculator,

$\therefore$ **The area of the region bounded by the curve and the x–axis between $x = \beta$ and $x = \gamma$**

A $= 0.688$ (3dp) **(ans)** [2]

(iii) **Using a non–calculator method,**

$\therefore$ Let B be the area of the shaded region bounded by the curve and the line.



First, locate the intersection point P.

❶: $y = x^3 - 3x + 1 = 1$

$$x^3 - 3x = 0 \quad \Rightarrow \quad x\left(x^2 - 3\right) = 0$$

$$\Rightarrow \; x = 0, \; x = \pm\sqrt{3}$$

The intersection point P $\equiv \left(-\sqrt{3}, 1\right)$

$\therefore$ **The area of the shaded region bounded by the curve and the line**

$$B = \int_P^Q y_1 - y_2 \, \mathrm{d}x$$

$$= \int_{-\sqrt{3}}^{0} x^3 - 3x + 1 - 1 \cdot \mathrm{d}x$$

$$= \int_{-\sqrt{3}}^{0} x^3 - 3x \cdot \mathrm{d}x$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{3}{2}x^2\right]_{-\sqrt{3}}^{0}$$

$$= -\left[\tfrac{1}{4}\left(-\sqrt{3}\right)^4 - \tfrac{3}{2}\left(-\sqrt{3}\right)^2\right]$$

$$= -\left[\tfrac{1}{4} \times 9 - \tfrac{3}{2} \times 3\right]$$

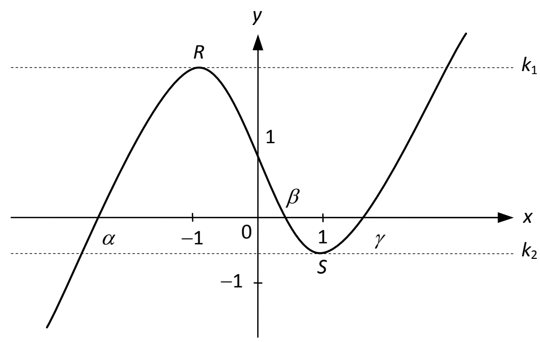
$= \frac{9}{4}$ units^2 **(ans)** [4]

☺ **Exam Report**

Most candidates gave the correct answer.

(iv) Given that $x^3 - 3x + 1 = k$ has three real distinct roots. — ❷

$\therefore$ To deduce the set of values of k that will satisfy ❷:



In order for ❷ to have 3 distinct roots, hence by shifting the line $y = k$, deduce that the limiting values for k would be k_1 and k_2 respectively.

Point R: When $x = -1$, $y = (-1)^3 - 3(-1) + 1$
$= 3$

Point S: When $x = 1$, $y = (1)^3 - 3(1) + 1$
$= -1$

$\therefore$ **The set of values of k that will give $x^3 - 3x + 1 = k$ three real distinct roots**

$\{k \in \mathbb{R} : -1 < k < 3\}$ **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer.

§

7. [Calculus]

Method

Approach I – differential equations

During early summer, John takes a milk bottle containing milk from a refrigerator and placed it on a table. The room temperature is at a constant 20 °C.

John found that as the milk warms up, the rate of increase of its temperature θ °C after time t minutes is proportional to the temperature difference (20 – θ) °C.

Given that initially the temperature of the milk is 10 °C and the rate of increase of the temperature is 1 °C per minute.

By setting up and solving a differential equation,

$\therefore$ **To show that $\vartheta = 20 - 10e^{-\frac{1}{10}t}$:**

Deduce that from the statement,

$$\frac{d\theta}{dt} \propto +(20-\theta)^\circ$$

$$\frac{d\theta}{dt} = k(20-\theta)^\circ$$, where k is a proportionality constant — ❶

Boundary conditions:

When θ= 10 °C, $\frac{d\theta}{dt} = 1$ °C per minute

❶: $1 = k(20-10)^\circ \Rightarrow 1 = k(10)^\circ$

$k = \frac{1}{10}$

Rearranging ❶,

$$\frac{d\theta}{(20-\theta)^\circ} = \frac{1}{10}\cdot dt$$

Integrating wrt t minutes,

$$\int \frac{d\theta}{(20-\theta)^\circ} = \int \frac{1}{10}\cdot dt$$

$-\ln(20-\theta)^\circ = \frac{1}{10}t + c$, where c is an integral constant

$$(20-\theta)^\circ = e^{-\left(\frac{1}{10}t+c\right)}$$

$$\theta^\circ = 20 - e^{-\left(\frac{1}{10}t+c\right)}$$

$$\theta^\circ = 20 - e^{-\left(\frac{1}{10}t+c\right)}$$

$$\theta^\circ = 20 - e^c\cdot e^{-\frac{1}{10}t}$$ — ❷

Boundary conditions:

When $t = 0$ minute, $\theta = 10$ °C

❷: $10^\circ = 20 - e^c\cdot e^{-\frac{1}{10}(0)} \Rightarrow 10 = 20 - e^c\cdot 1$

$e^c = 10$

Rewriting ❷:

$\theta^\circ = 20 - 10\cdot e^{-\frac{1}{10}t}$ **(QED)** [7] — ❸

☺ **Exam Report**

Most candidates gave the correct answer. But, there were several scripts that gave labourious algebra and wrong differential equation.

$\therefore$ Let T be the time taken for the milk to reach a temperature of 15 °C.

❸: $\theta^\circ = 20 - 10\cdot e^{-\frac{1}{10}t}$

$15 = 20 - 10\cdot e^{-\frac{1}{10}T} \Rightarrow 10\cdot e^{-\frac{1}{10}T} = 20 - 15 \Rightarrow$

$10\cdot e^{-\frac{1}{10}T} = 5 \Rightarrow e^{-\frac{1}{10}T} = \frac{5}{10} \Rightarrow 10\cdot e^{-\frac{1}{10}T} = 5$

$\Rightarrow -\frac{1}{10}T = -\ln 2 \Rightarrow \frac{1}{10}T = \ln 2 \Rightarrow T = 10\cdot\ln 2$

$\therefore$ **The time taken for the milk to reach a temperature of 15 °C**

$T = 10\cdot\ln 2$ minutes **(ans)**

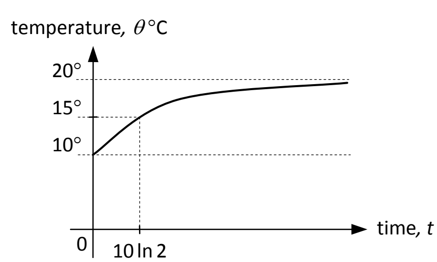
$\therefore$ **For large values of time t minutes,**

From ❸:

$-10\cdot e^{-\frac{1}{10}t} \to 0$

$\Rightarrow \theta \to 20$ °C (room temperature) **(ans)**

$\therefore$ **A graph of θ against t** is sketched.



(ans) [4]

☺ **Newton's law of cooling**

This is a classical standard case.

§

8. [Complex Numbers]

Method

Approach I – loci

On a complex plane, two complex numbers z_1 and z_2 are defined as

$1+\mathrm{i}\sqrt{3}$ and $-1-\mathrm{i}$ respectively.

(i) [For this question, give answer in exact form.]

$\therefore$ To express each of z_1 and z_2 in polar form

$r(\cos\vartheta + \mathrm{i}\sin\vartheta)$, where $r > 0$ and $-\pi < \theta \le \pi$:

$$z_1 = \sqrt{1^2+\sqrt{3}^2}\left(\cos\left(\tan^{-1}\frac{\sqrt{3}}{1}\right)+\mathrm{i}\sin\left(\tan^{-1}\frac{\sqrt{3}}{1}\right)\right)$$

$$= 2\left(\cos\left(\tfrac{1}{3}\pi\right)+\mathrm{i}\sin\left(\tfrac{1}{3}\pi\right)\right) \quad \textbf{(ans)}$$

$$z_2 = \sqrt{(-1)^2+(-1)^2}\left(\cos\left(\tan^{-1}\frac{-1}{-1}\right)+\mathrm{i}\sin\left(\tan^{-1}\frac{-1}{-1}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(-\tfrac{3}{4}\pi\right)+\mathrm{i}\sin\left(-\tfrac{3}{4}\pi\right)\right) \quad \textbf{(ans)} \quad [2]$$

☺ **CheckBack**

- If the answer is $2\left(\cos\left(\tfrac{1}{3}\pi\right)+\mathrm{i}\sin\left(\tfrac{1}{3}\pi\right)\right)$,

  $z_1 = 2\cos\left(\tfrac{1}{3}\pi\right)+2\mathrm{i}\sin\left(\tfrac{1}{3}\pi\right)$

  $= 2\times\tfrac{1}{2}+\mathrm{i}\cdot 2\times\tfrac{\sqrt{3}}{2} = 1+\mathrm{i}\sqrt{3}$ **(checked)**

- If the answer is $2\left(\cos\left(\tfrac{1}{3}\pi\right)+\mathrm{i}\sin\left(\tfrac{1}{3}\pi\right)\right)$,

  $z_2 = \sqrt{2}\left(\cos\left(-\tfrac{3}{4}\pi\right)+\mathrm{i}\sin\left(-\tfrac{3}{4}\pi\right)\right)$

  $= \sqrt{2}\times\left(-\tfrac{1}{\sqrt{2}}\right)+\sqrt{2}\cdot\mathrm{i}\cdot\left(-\tfrac{1}{\sqrt{2}}\right)$

  $= -1-\mathrm{i}$ **(checked)**

(ii) [For this question, give answer in exact polar form.]

$$\therefore \frac{z_1}{z_2} = \frac{2\,\mathrm{cis}\left(\tfrac{1}{3}\pi\right)}{\sqrt{2}\,\mathrm{cis}\left(-\tfrac{3}{4}\pi\right)}$$

$$= \frac{2\,\mathrm{cis}\left(\tfrac{1}{3}\pi\right)}{\sqrt{2}\,\mathrm{cis}\left(\tfrac{5}{4}\pi\right)} = \sqrt{2}\,\mathrm{cis}\left(\tfrac{1}{3}\pi-\left(\tfrac{5}{4}\pi\right)\right)$$

$$= \sqrt{2}\,\mathrm{cis}\left(-\tfrac{11}{12}\pi\right)$$

$$= \sqrt{2}\,\mathrm{cis}\left(\tfrac{13}{12}\pi\right)$$

$\therefore$ The complex conjugate of $\frac{z_1}{z_2}$

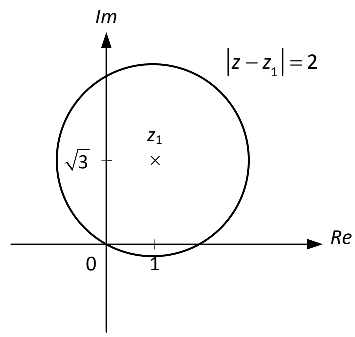
$$= \frac{z_1}{z_2}^{*} = \sqrt{2}\,\mathrm{cis}\left(\tfrac{11}{12}\pi\right)$$

$$= \sqrt{2}\left(\cos\left(\tfrac{11}{12}\pi\right)+\mathrm{i}\sin\left(\tfrac{11}{12}\pi\right)\right) \quad \textbf{(ans)} \quad [3]$$

(iii) **On a single Argand diagram, the sketch of the loci**

(a) $|z-z_1| = 2$,

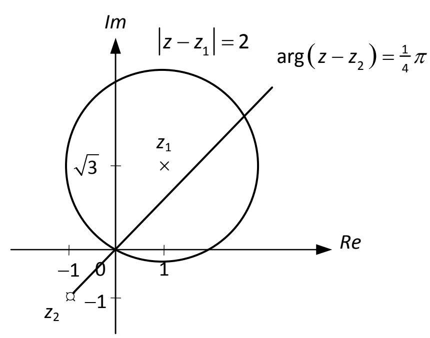
$\therefore$ A circle of radius 2 units and centered at point z_1.



(ans)

(b) $\arg(z-z_2) = \frac{1}{4}\pi$

$\therefore$ A half line centered at point z_2 at 45° to the horizontal *real axis*. It does not include the point z_2.



(ans) [4]

☺ **Exam Report**

Most candidates gave the correct answer. Sketches that included the point z_2 were not credited in full.

(iv) The locus $|z - z_1| = 2$ meets the positive real axis.

$\therefore$ Let x_1 be the x–value where the locus $|z - z_1| = 2$ meets the positive *real axis*.

Express $|z - z_1| = 2$ in Cartesian form,

$$\left(y-\sqrt{3}\right)^2 + (x-1)^2 = 2^2 \qquad \text{— ❶}$$

When $y = 0$,

❶: $\left(0-\sqrt{3}\right)^2 + (x-1)^2 = 2^2$

$\Rightarrow \; 3 + (x-1)^2 = 4 \;\Rightarrow\; (x-1)^2 = 1$

$\Rightarrow \; (x-1) = \pm 1 \;\Rightarrow\; x = 1 \pm 1 \;\Rightarrow\; x = 0, 2$

$\therefore$ **The x–value where the locus $|z - z_1| = 2$ meets the positive *real axis***

$x_1 = 2$ **(ans)** [2]

☺ **Exam Report**

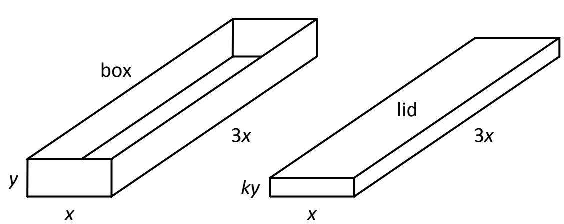
Most candidates gave the correct answer. The sketch for the graph was badly drawn. Marks were not awarded in many cases.

§

9. [Calculus]

Method

Approach I – maxima & minima



A manufacturing company requires to make a powder box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The diagram shows the proposed box. It is given that the length of the box is $3x$ cm, the width is x cm and the height is y cm. The lid fits the box exactly and has depth ky cm, where $0 < k \le 1$.

(i) **Using differentiation, in terms of k,**

$\therefore$ Let x_{min} be the value of x which gives a minimum total external surface area of the box and the lid.

Total volume $= x \times 3x \times y = 3x^2y = 300$ — ❶

Total Area, $A = 2(xy) + 2(3xy) + (3x^2)$ — box

$+ 2(kxy) + 2(3kxy) + (3x^2)$ — lid

$= 2xy + 6xy + 3x^2 + 2kxy + 6kxy + 3x^2$

$= 8xy + 6x^2 + 8kxy$ — ❷

Substituting y from ❶ into ❷:

$$A = \; A = 8x\left(\frac{300}{3x^2}\right) + 6x^2 + 8kx\left(\frac{300}{3x^2}\right)$$

$$= \frac{800}{x} + 6x^2 + \frac{800k}{x} \qquad \text{— ❸}$$

Differentiating ❸ wrt x,

$$\frac{dA}{dx} = -\frac{800}{x^2} + 12x - \frac{800k}{x^2} \qquad \text{— ❹}$$

For turning point,

$$\frac{dA}{dx} = 0 = -\frac{800}{x^2} + 12x - \frac{800k}{x^2}$$

$\Rightarrow \; 0 = -800 + 12x^3 - 800k$

$\Rightarrow \; 12x^3 = 800 + 800k \;\Rightarrow\; x = \sqrt[3]{\dfrac{800+800k}{12}}$

$\Rightarrow \; x = \sqrt[3]{\dfrac{200(1+k)}{3}}$

Differentiating ❹ wrt x,

$$\frac{d^2A}{dx^2} = \frac{1600}{x^3} + 12 + \frac{1600k}{x^3} \qquad \text{— ❺}$$

When $x = \sqrt[3]{\dfrac{200(1+k)}{3}}$, ❺ is always positive.

$\Rightarrow \; x = \sqrt[3]{\dfrac{200(1+k)}{3}}$ is a minimum.

$\therefore$ **The value of x which gives a minimum total external surface area of the box and the lid**

$x_{min} = \sqrt[3]{\dfrac{200(1+k)}{3}}$ **(ans)** [6]

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer. However, there were many careless algebra which the answer marks were not credited in many cases.

(ii) [For this question, give answer in its simpliest form.]

$\therefore$ **The ratio of the height to the width, $\frac{y}{x}$, in this case**

$$= \frac{\left(\frac{300}{3x^2}\right)}{x} = \frac{100}{x^3} = \frac{100}{{x_{\min}}^3} = \frac{100}{\left(\sqrt[3]{\frac{200(1+k)}{3}}\right)^3}$$

$$= \frac{100}{\frac{200(1+k)}{3}}$$

$$= \frac{3}{2(1+k)}$$ **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer, provided the initial answer to part (i) was correct.

(iii) **As k varies,**

$\therefore$ Maximum value of k is 1.

Minimum value of k is 0.

$\therefore$ **The values between which $\frac{y}{x}$ must lie**

$$\frac{3}{2(1+1)} \le \frac{y}{x} \le \frac{3}{2(1+0)}$$

$$\Rightarrow \frac{3}{4} \le \frac{y}{x} \le \frac{3}{2}$$ **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer, provided the initial answer to part (ii) was correct.

(iv) **The box has square ends.**

$\therefore$ This implies that $y = x$.

$\therefore$ **The value of k, in which the box has square ends**

$$\Rightarrow \frac{y}{x} = \frac{3}{2(1+k)} = 1$$

$$\Rightarrow \frac{3}{2} = (1+k) \Rightarrow k = \tfrac{3}{2} - 1$$

$$\Rightarrow k = \tfrac{1}{2}$$ **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

Most candidates gave the correct answer, provided the initial answer to part (ii) was correct.

§

10. [Vectors]

Method

Approach I – planes

In a vector space, a line ℓ is defined by

$$\ell: \frac{x-10}{-3} = \frac{y+1}{6} = \frac{z+3}{9},$$

and a plane p is defined by

$$p: x - 2y - 3z = 0.$$

(i) $\therefore$ **To show that ℓ is perpendicular to p:**

Rewriting the equations in unit vector form:

$$\ell: \mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix} \Rightarrow \mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

where μ is a scalar variable

$$p: \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 0 \quad \text{—❶}$$

Since the *direction vector* of the line ℓ is parallel to the *perpendicular vector* of the plane, p, it implies that the line ℓ is perpendicular to the plane, p. **(QED)** [2]

(ii) $\therefore$ Let C be the coordinates of the point of intersection of ℓ and the plane, p.

The point of intersection must satisfy both line and plane equations.

$$\Rightarrow \left(\begin{pmatrix}10\\-1\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$$

$$\Rightarrow \begin{pmatrix}10-\mu\\-1+2\mu\\-3+3\mu\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$$

$$\Rightarrow -(10-\mu)+2(-1+2\mu)+3(-3+3\mu)=0$$

$$\Rightarrow -10+\mu-2+4\mu-9+9\mu=0$$

$$\Rightarrow -21+14\mu=0 \quad \Rightarrow \quad \mu=\tfrac{3}{2}$$

$\therefore$ The coordinates of the point of intersection of ℓ and the plane, p

$$C=\begin{pmatrix}10\\-1\\-3\end{pmatrix}+\tfrac{3}{2}\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}10\\-1\\-3\end{pmatrix}+\begin{pmatrix}-\frac{3}{2}\\3\\\frac{9}{2}\end{pmatrix}$$

$$=\begin{pmatrix}8\frac{1}{2}\\2\\1\frac{1}{2}\end{pmatrix}\equiv\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)\ (F)$$ **(ans)** [4]

☺ **CheckBack**

If the answer is $\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$,

Substituting into ❶:

$$\text{LHS}=\begin{pmatrix}8\frac{1}{2}\\2\\1\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}=-8\tfrac{1}{2}+4+\tfrac{9}{2}=0$$

= RHS (a point on the plane) **(checked)**

☺ **Exam Report**

Many candidates gave the correct answer.

(iii) $\therefore$ **To show that the point A with coordinates (–2, 23, 33) lies on ℓ.**

Recall, $\mathbf{r}=\begin{pmatrix}10\\-1\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}$

Consider the x–value, $10-\mu=-2 \quad\Rightarrow\quad \mu=12$

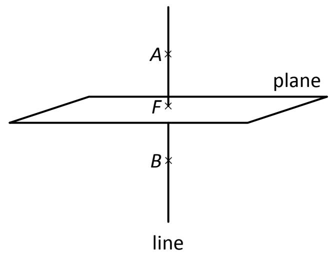
When $\mu=12$,

The coordinates of point $A=\begin{pmatrix}10\\-1\\-3\end{pmatrix}+12\begin{pmatrix}-1\\2\\3\end{pmatrix}$

$$=\begin{pmatrix}-2\\23\\33\end{pmatrix}$$

= (–2, 23, 33) $\Rightarrow$ lies on the line **(QED)**

$\therefore$ To find the coordinates of the point B which is the mirror image of point A in the plane, p.



For the point B to be the mirror image of point A (passing through the perpendicular line) in the plane, p

$\Rightarrow$ The step from point A to point F is the same as the step from point F to point B

$\Rightarrow$ point A $\rightarrow$ point F $\Rightarrow$ point F $\rightarrow$ point B

$\Rightarrow$ (–2, 23, 33) $\rightarrow$ $\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$

$\Rightarrow$ $\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$ $\rightarrow$ point B

$\Rightarrow$ (–2, 23, 33) $\rightarrow$ $\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$

$\equiv$ (+10½, –21, –31½)

$\Rightarrow$ $\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$

$\rightarrow$ point $B=\left(8\tfrac{1}{2},2,1\tfrac{1}{2}\right)$ + (+10½, –21, –31½)

= (19, –19, –30)

$\therefore$ **The coordinates of the point B which is the mirror image of point A in the plane, p**

= (19, –19, –30) **(ans)** [3]

☺ **CheckBack**

If the answer is (19, –19, –30),

The mid–point of AB = [(–2, 23, 33) + (19, –19, –30)] /2

= (8½, 2, 1½) ≡ point F **(checked)**

☺ **Exam Report**

Not that many candidates gave the correct answer. Many scripts related to employing complex algebra to acquire the point B, but the complexity and challenge of time caused many careless mistakes which did not lead to the correct answer.

(iv) [For this question, give answer to the nearest whole number.]

Approach I – cross product

$\therefore$ **The area of triangle *OAB*, where *O* is the origin**

$$= \tfrac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$$

$$= \tfrac{1}{2}\left|(-2, 23, 33)\times(19, -19, -30)\right|$$

$$= \tfrac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 23 & 33 \\ 19 & -19 & -30 \end{vmatrix}$$

$$= \tfrac{1}{2}\left|\left[\mathbf{i}\begin{vmatrix} 23 & 33 \\ -19 & -30 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 33 & -2 \\ -30 & 19 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 23 \\ 19 & -19 \end{vmatrix}\right]\right|$$

$$= \tfrac{1}{2}\left|\begin{bmatrix} \mathbf{i}\left[(23\times-30)-(-19\times33)\right] \\ +\mathbf{j}\left[(33\times19)-(-30\times-2)\right] \\ +\mathbf{k}\left[(-2\times-19)-(19\times23)\right] \end{bmatrix}\right|$$

$$= \tfrac{1}{2}\left|\mathbf{i}[-63]+\mathbf{j}[567]+\mathbf{k}[-399]\right|$$

$$= \tfrac{1}{2}\left|\sqrt{[-63]^2+[567]^2+[-399]^2}\right|$$

$= ½\ (696.174547)$

≈ 348 (whole number) units^2 **(ans)** [3]

Approach II – trigonometry

$\therefore$ Let T be the area of triangle OAB, where O is the origin

$$\left|\overrightarrow{OF}\right| = \sqrt{8\tfrac{1}{2}^2+2^2+1\tfrac{1}{2}^2}$$

$= 8.86$

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{OB}-\overrightarrow{OA}\right| = \left|(19, -19, -30)-(-2, 23, 33)\right|$$

$$= \left|(21, -42, -63)\right| = \left|\sqrt{21^2+42^2+63^2}\right|$$

$= 78.5748$

$\therefore$ **The area of triangle *OAB*, where *O* is the origin**

$T = ½ \times$ base $\times$ height

$= ½ \times 78.5748 \times 8.86 = 348.086364$

$= 348$ units^2 **(ans)** [3]

☺ **CheckBack**

If the answer is 348 units^2,

There is no easy CheckBack option.

One could consider finding the area using the other approach. **(checked)**

☺ **Exam Report**

This question was poorly answered. Many candidates approached the question using very long–winged method.

Candidates should be able to employ the following dot–product approach:

$$\text{Area} = \tfrac{1}{2}\sqrt{\left|\overrightarrow{OA}\right|^2\left|\overrightarrow{OB}\right|^2-\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)^2}$$

§

11. [Functions & Graphs]

Method

Approach I – sketching

A curve C has a pair of parametric equations defined as

$$x = t+\frac{1}{t} \qquad — ❶$$

$$y = t-\frac{1}{t} \qquad — ❷$$

(i) **If a point *P* on the curve has parameter *p*,**

$\therefore$ **To show that the equation of the tangent at *P* is**

$$\left(p^2+1\right)x-\left(p^2-1\right)y = 4p$$

Applying the parameter p, the point P has

❶: $x_p = p+\dfrac{1}{p}$

❷: $y_p = p-\dfrac{1}{p}$

Differentiating ❶ and ❷ wrt t:

❶: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1-\dfrac{1}{t^2}$

❷: $\dfrac{dy}{dt}=1+\dfrac{1}{t^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt} \Rightarrow \frac{dy}{dx}=1+\frac{1}{t^2}\Big/1-\frac{1}{t^2}$$

Applying the parameter p,

$$\frac{dy}{dx}=1+\frac{1}{p^2}\Big/1-\frac{1}{p^2}$$

Equation of tangent at point P:

$$\frac{y-\left(p-\frac{1}{p}\right)}{x-\left(p+\frac{1}{p}\right)}=1+\frac{1}{p^2}\Big/1-\frac{1}{p^2}$$

$$\Rightarrow \left(1-\frac{1}{p^2}\right)y-\left(1-\frac{1}{p^2}\right)\left(p-\frac{1}{p}\right)=\left(1+\frac{1}{p^2}\right)x-\left(1+\frac{1}{p^2}\right)\left(p+\frac{1}{p}\right)$$

$$\Rightarrow \left(p^2-1\right)y-\left(p^2-1\right)\left(p-\frac{1}{p}\right)=\left(p^2+1\right)x-\left(p^2+1\right)\left(p+\frac{1}{p}\right)$$

$$\Rightarrow \left(p^2+1\right)\left(p+\frac{1}{p}\right)-\left(p^2-1\right)\left(p-\frac{1}{p}\right)=\left(p^2+1\right)x-\left(p^2-1\right)y$$

$$\Rightarrow p^3+p+p+\frac{1}{p}-p^3+p+p-\frac{1}{p}=\left(p^2+1\right)x-\left(p^2-1\right)y$$

$$\Rightarrow 4p=\left(p^2+1\right)x-\left(p^2-1\right)y \quad \textbf{(QED)} \quad [4] \quad ❸$$

☺ **Exam Report**

Most candidates did not find this proof difficult. Some algebra expressions did look strange.

(ii) The tangent at P meets the line $y=x$ at the point A and the line $y=-x$ at the point B.

∴ **To show that the area of triangle OAB is independent of p, where O is the origin.**

From ❸:

Point A: $4p=\left(p^2+1\right)x-\left(p^2-1\right)x$

$\Rightarrow 4p=2x \Rightarrow x=2p$

$\Rightarrow (2p, 2p)$

Point B: $4p=\left(p^2+1\right)x-\left(p^2-1\right)(-x)$

$\Rightarrow 4p=2p^2x \Rightarrow x=\dfrac{2}{p}$

$\Rightarrow \left(\dfrac{2}{p},-\dfrac{2}{p}\right)$

Area of the triangle OAB

$$=\tfrac{1}{2}\sqrt{\left|\overrightarrow{OA}\right|^2\left|\overrightarrow{OB}\right|^2-\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)^2}$$

$$=\tfrac{1}{2}\sqrt{\left|4p^2+4p^2\right|\left|\frac{4}{p^2}+\frac{4}{p^2}\right|-\left(\overrightarrow{OA}\cdot\overrightarrow{OB}\right)^2}$$

$$=\tfrac{1}{2}\sqrt{8\times 8-\left[(2p,2p)\cdot\left(\frac{2}{p},-\frac{2}{p}\right)\right]^2}$$

$$=\tfrac{1}{2}\sqrt{64-[4-4]^2}$$

$= 4$ units^2 (independent of p) **(QED)** [4]

(iii) ∴ To find a Cartesian equation of C.

Squaring both ❶ & ❷:

❶^2: $x^2=\left(t+\dfrac{1}{t}\right)^2=t^2+2+\dfrac{1}{t^2}$

❷^2: $y^2=\left(t-\dfrac{1}{t}\right)^2=t^2-2+\dfrac{1}{t^2}$

❶^2 – ❷^2:

$x^2-y^2=4$

∴ **The Cartesian equation of C**

$x^2-y^2=4$ **(ans)**

☺ **CheckBack**

There is no easy CheckBack option for this question.

☺ **Exam Report**

An unusual number of candidates did not get this question correct.

∴ **The sketch of the curve C:**

$x^2-y^2=4$ —— ❹

Rearranging ❹,

$y=\pm\sqrt{x^2-4}$

Locating the coordinates of any points where the curve C crosses the x – and y –axes and the equations of any asymptotes.

- *Crossings*

  When $x=0$, $y=$ undefined (*i.e.*, the curve never cut the *y*–axis.)

  When $y=0$, $x=$ $x=\pm 2$

- *Extreme values*
  - o When $x\to -\infty$, $y\to \pm x$.
  - o When $x\to +\infty$, $y\to \pm x$.

- *Asymptotes*

  $y=\pm x$

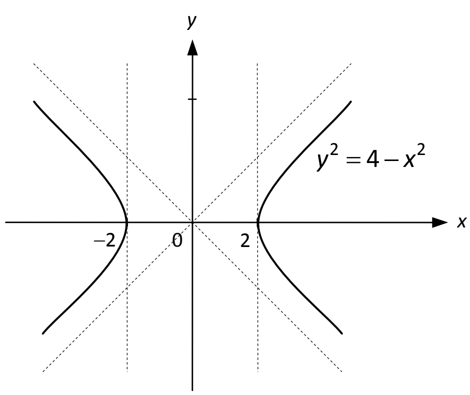
- *Symmetry*

  The curve is symmetrical about *y*–axis and the *x*–axis.

- *Turning points*

  From $\dfrac{\mathrm{dy}}{\mathrm{dx}}=1+\dfrac{1}{t^2}\Big/1-\dfrac{1}{t^2}$, the curve has no stationary turning points.

- *Sketch*



(ans) [4]

☺ **General curve sketching**

To sketch a given unfamiliar equation of graph, certain characteristics must be obtained in order to sketch it:

❶ Symmetry,

❷ Intersections with the axes,

❸ Turning points,

❹ Asymptotes,

❺ Axes of symmetry, and

❻ Restrictions on the possible values of *x* and/or *y*

☺ **Exam Report**

Apart from the difficulty in getting the equation of the curve, most candidates gave the correct sketch of the graph, if they derived the equation of curve correctly.

§

Learning objectives implemented in 2010 Part 2 9740/2

Questions

Section A: Pure Mathematics [40 marks]

Answer all questions.

1. [Complex Numbers]

Method

Approach I – complex roots

(i) The function is defined as

$$x^2 - 6x + 34 = 0$$

$\therefore$ **To find the value of x:**

Recall,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$= x = \frac{6 \pm \sqrt{6^2 - 4(34)}}{2} = \frac{6 \pm \sqrt{36 - 136}}{2}$$

$$= \frac{6 \pm \sqrt{-100}}{2}$$

$= 3 \pm 5\text{i}$ **(ans)** [2]

☺ **CheckBack**

If the answer is $3 \pm 5\text{i}$,

The function is

$$(x - 3 - 5\text{i})(x - 3 + 5\text{i}) = 0$$
$$x^2 - 3x - 5\text{i}x - 3x + 9 + 15\text{i} + 5\text{i}x - 15\text{i} + 25 = 0$$
$x^2 - 6x + 34 = 0$ **(checked)**

☺ **Exam Report**

Almost all candidates gave the correct answer.

(ii) The function is defined as

$x^4 + 4x^3 + x^2 + ax + b = 0$, where a and b are real — ❶

Given that one of its roots is $x = -2 + \text{i}$.

$\therefore$ **To find the values of a and b:**

Since $x = -2 + \text{i}$ is a root, its conjugate is also a root, *i.e.*, $x = -2 - \text{i}$

The factor is therefore,

$$(x + 2 - \text{i})(x + 2 + \text{i})$$
$$\Rightarrow (x^2 + 2x - \text{i}x + 2x + 4 - 2\text{i} + \text{i}x + 2\text{i} + 1)$$
$$\Rightarrow (x^2 + 4x + 5)$$

Using long division,

$$\begin{array}{r|l} & x^2 - 4 \\ \hline x^2 + 4x + 5 & x^4 + 4x^3 + x^2 + ax + b \\ & \underline{x^4 + 4x^3 + 5x^2} \\ & \qquad -4x^2 + ax + b \\ & \qquad \underline{-4x^2 - 16x - 20} \\ & \qquad (a+16)x + (b+20) \end{array}$$

Since $(x^2 + 4x + 5)$ is a factor of ❶, it will have no remainder,

i.e., $a = -16$ and $b = -20$ **(ans)**

$\therefore$ **To find the other roots:**

Deduce from the long division that

$(x^2 - 4)$ is the remaining factor — ❷

Simplifying ❷ further,

$$(x - 2)(x + 2)$$

$\therefore$ **The other roots of equation ❶**

i.e., $x = -2 - \text{i}$ and ± 2 **(ans)** [5]

☺ **CheckBack**

If the answer is ± 2,

$$f(+2) = (2)^4 + 4(2)^3 + (2)^2 - 16(2) - 20 = 0$$

$$f(-2) = (-2)^4 + 4(-2)^3 + (-2)^2 - 16(-2) - 20 = 0$$

(checked)

☺ **Exam Report**

Almost all candidates gave the correct answer.

Candidates who did not quote one of the factors as $x = -2 - \mathrm{i}$ was deducted one final answer mark.

§

2. [Sequences & Series]

Method

Approach I – mathematical induction

(i) The summation of a series is defined as

$$\sum_{r=1}^{n} r(r+2) = \tfrac{1}{6}n(n+1)(2n+7)$$

$\therefore$ **Proof by mathematical induction:**

$\therefore$ Let P_n be

$$\sum_{r=1}^{n} r(r+2) = \tfrac{1}{6}n(n+1)(2n+7) \quad \text{where } n \in \mathbb{Z}^+$$

When $n = 1$,

$$\text{LHS of } P_1 = \sum_{r=1}^{1} r(r+2) = 1(1+2) = 3$$

RHS of P_1

$$= \tfrac{1}{6}n(n+1)(2n+7) = \tfrac{1}{6}(1)(1+1)(2(1)+7)$$

$$= 3 = \text{LHS of } P_1$$

$\therefore$ P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, *i.e.*,

$$\sum_{r=1}^{n=k} r(r+2) = \tfrac{1}{6}k(k+1)(2k+7).$$

When $n = k+1$,

$$\text{RHS of } P_{k+1} = \tfrac{1}{6}(k+1)(k+2)(2k+9)$$

$$\text{LHS of } P_{k+1} = \sum_{r=1}^{n=k+1} r(r+2)$$

$$= \sum_{r=1}^{n=k} r(r+2) + (k+1)(k+1+2)$$

$$= \tfrac{1}{6}k(k+1)(2k+7) + (k+1)(k+1+2)$$

$$= \tfrac{1}{6}(k+1)\left[k(2k+7) + 6(k+3)\right]$$

$$= \tfrac{1}{6}(k+1)\left[2k^2 + 7k + 6k + 18\right]$$

$$= \tfrac{1}{6}(k+1)\left[2k^2 + 13k + 18\right]$$

$$= \tfrac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \text{RHS of } P_{k+1}$$

$\therefore$ P_{k+1} is true when P_k is true.

$\therefore$ By **mathematical induction**, P_n is true for all $n \in \mathbb{Z}^+$. **(QED)** [5]

☺ **Mathematic induction**

Mathematical induction refers to the method which is commonly used to prove mathematical statements, which refer to a variable natural number.

Steps for mathematical induction:

❶ Let P_n be the statement about a variable natural number n.

❷ Prove that the statement is true for m where m is the least value that n can assume, usually $m = 1$, *i.e.*, show that P_m is true.

❸ Assume that the statement is true for $n = k$ and show that the statement is also true for $n = k + 1$, *i.e.*, show that P_k is true $\Rightarrow P_{k+1}$ is true.

❹ Combine ❷ and ❸, $P_m \Rightarrow P_{m+1} \Rightarrow P_{m+2} \Rightarrow \ldots$, and conclude that "By mathematical induction, the statement is true for all $n \in Z^+, n \geq m$."

☺ **Exam Report**

A candidate suggested the conclusion to proof of the mathematical induction as "Since P_1 is true and P_k is true implies P_{k+1} is true, P_n is true by mathematical induction" is clearly illogical.

If P_k was already true and $k \in \mathbb{Z}^+$, then there would be no further need to prove anymore.

The proof must clearly state that when P_k is true, P_{k+1} is proved to be true. Since P_1 is proved to be true, it implies P_2 is true. Since P_2 is true, it implies P_3 is also true, and so on. The above statement given by the candidate is clearly on the wrong logical presentation. Another essential statement in the final description must be that "P_n is true for all $n \in \mathbb{Z}^+$".

Prove by mathematical induction is a very strict proof, several marks or the final answer mark were not credited.

(ii)
(a) The summation of a series is defined as

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \tfrac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \quad — ❶$$

∴ **Proof by method of differences:**

Consider splitting the LH term into its partial fractions,

$$\frac{1}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2} = \frac{A(r+2)+Br}{r(r+2)}$$

$$= \frac{(A+B)r+2A}{r(r+2)} \Rightarrow A = ½ \text{ and } B = -½$$

LHS of ❶ $= \sum_{r=1}^{n} \frac{1}{r(r+2)}$

$$= \sum_{r=1}^{n} \tfrac{1}{2}\left[\frac{1}{r} - \frac{1}{r+2}\right] = \tfrac{1}{2}\sum_{r=1}^{n}\left[\frac{1}{r} - \frac{1}{r+2}\right]$$

$$= \tfrac{1}{2}\left[\begin{array}{l}\left(\frac{1}{1}-\frac{1}{1+2}\right)+\left(\frac{1}{2}-\frac{1}{2+2}\right)+\left(\frac{1}{3}-\frac{1}{3+2}\right)+\left(\frac{1}{4}-\frac{1}{4+2}\right)+\ldots \\ +\left(\frac{1}{n-1}-\frac{1}{n-1+2}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\end{array}\right]$$

$$= \tfrac{1}{2}\left[\begin{array}{l}\left(\frac{1}{1}-\cancel{\frac{1}{1+2}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{2+2}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{3+2}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{4+2}}\right)+\ldots \\ +\left(\cancel{\frac{1}{n-2}}-\cancel{\frac{1}{n-2+2}}\right)+\left(\cancel{\frac{1}{n-1}}-\frac{1}{n-1+2}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)\end{array}\right]$$

$$= \tfrac{1}{2}\left[\left(\frac{1}{1}\right)+\left(\frac{1}{2}\right)+\left(-\frac{1}{n+1}\right)+\left(-\frac{1}{n+2}\right)\right]$$

$$= \tfrac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

= RHS of ❶ **(QED)** [4]

☺ **Exam Report**

This was deemed a straight–forward question and most candidates gave the correct answer.

(b) ∴ **The explanation why $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ is a convergent series:**

As n → ∞, the terms $\frac{1}{2(n+1)}$ and $\frac{1}{2(n+2)}$ becomes negligibly small and therefore can be ignored (from ❶).

Hence, as n → ∞,

$$\sum_{r=1}^{\infty} \frac{1}{r(r+2)} \rightarrow ¾ \text{ (convergent series)} \quad \textbf{(QED)}$$

∴ **The value of the sum to infinity**

$$\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \tfrac{3}{4} \quad \textbf{(ans)} \quad [2]$$

☺ **Exam Report**

Most candidates gave the correct answer.

§

3. [Functions & Graphs]

Method

Approach I – graph

(i) [For this question, give your answer as a single algebraic fraction.]

The curve is defined as

$$y = x\sqrt{x+2} \quad — ❶$$

∴ **Differentiating ❶ wrt *x*:**

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \sqrt{x+2} + \frac{x\left(\frac{1}{2}\right)}{\sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+2}}\left[x+2+\tfrac{1}{2}x\right]$$

$$= \frac{1}{\sqrt{x+2}}\left[\tfrac{3}{2}x+2\right] \quad \textbf{(ans)}$$

$\therefore$ **Hence, to show that there is only one value of x for which the curve $y = x\sqrt{x+2}$ has a turning point.**

For stationary turning point,

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}\left[\tfrac{3}{2}x+2\right] = 0$$

$\Rightarrow$ $x = -\frac{4}{3}$ (only one point (x–value) is possible) **(QED)**

$\therefore$ **The one value of x for which the curve $y = x\sqrt{x+2}$ has a turning point**

$x = -\frac{4}{3}$ **(ans)** [5]

☺ **Exam Report**

Most candidates gave the correct answer. Some careless candidates gave the answer as $x = \frac{4}{3}$.

(ii) A curve is defined as

$$y^2 = x^2(x+2) \qquad \text{— ❷}$$

(a) [For this question, give answer in exact form.]

$\therefore$ Let g be the possible values of the gradient at the point where $x = 0$.

Differentiating ❷ wrt x:

$$2y\frac{dy}{dx} = 2x(x+2) + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(x+2)+x^2}{\pm 2\sqrt{x^2(x+2)}}$$

$$= \pm\left[\sqrt{(x+2)} + \frac{\frac{1}{2}x}{\sqrt{(x+2)}}\right]$$

$\therefore$ **Where $x = 0$,**

The possible values of the gradient

$$= \pm\left[\sqrt{(0+2)} + \frac{\frac{1}{2}(0)}{\sqrt{(0+2)}}\right]$$

$= \pm\sqrt{2}$ **(ans)** [2]

(b) $\therefore$ **The sketch of the curve $y^2 = x^2(x+2)$:**

- *Crossings*

  When $x = 0$, $y = 0$

  When $y = 0$, $x = 0, -2$

- *Extreme values*

  o When $x \to -\infty$, $y =$ undefined.

  o When $x \to +\infty$, $y \to \pm\infty$.

- *Turning point(s)*

  The curve has one stationary turning point at $x = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}\sqrt{-\frac{4}{3}+2} = -\frac{4}{3}\sqrt{\frac{2}{3}} \approx -1.09$

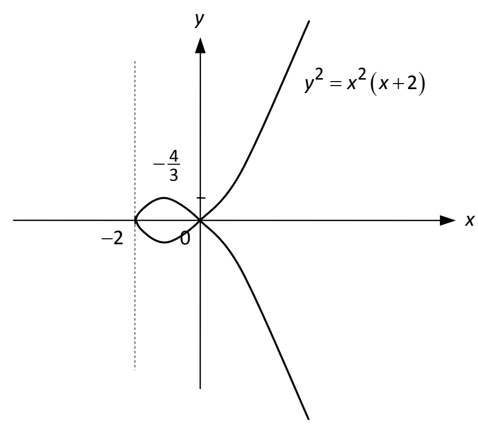
- *Asymptote(s)*

  There are no real asymptotes, except $x = -2$ and $y = x$ are approximate asymptotes.

- *Symmetry*

  The curve is symmetrical about x–axis.

- *Sketch*



Please check against the graphic calculator.

(ans) [2]

(iii) **On a separate diagram,**

$\therefore$ **The sketch of $y = f'(x)$, where $f(x) = x\sqrt{x+2}$:**

$$y = f'(x) \quad \Rightarrow \quad y' = \frac{1}{\sqrt{x+2}}\left[\tfrac{3}{2}x + 2\right]$$

- *Crossings*

  When $x = 0$, $y = \sqrt{2}$

  When $y = 0$, $x = x = -\frac{4}{3}$

- *Extreme values*
  - o When $x \to -\infty$, $y =$ undefined.
  - o When $x \to +\infty$, $y \to \pm\infty$.

- *Asymptote(s)*

  $x = -2$

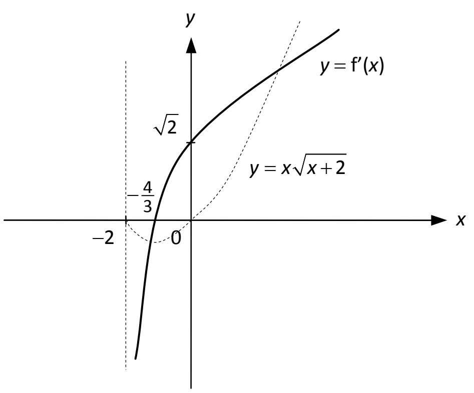
- *Turning point(s)*

  The curve has no obvious turning points.

- *Symmetry*

  The curve is not symmetrical.

- *Sketch*



$\therefore$ **The equation of asymptote:**

$x = -2$ **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer using their graphic calculator.

There were scripts that erroneously presented the graph of $f^{-1}(x)$ instead.

§

4. [Functions & Graphs]

Method

Approach I – composite functions

The function f is defined as

$$f: x \mapsto \frac{1}{x^2 - 1} \quad \text{for} \quad x \in \mathbb{R}, \quad x \neq \pm 1$$

(i) $\therefore$ **The sketch of the graph of $y = f(x)$:**

$$y = f(x) = \frac{1}{x^2 - 1}$$

- *Crossings*

  When $x = 0$, $y = -1$

  When $y = 0$, $x =$ undefined

- *Extreme values*
  - o When $x \to -\infty$, $y \to 0$.
  - o When $x \to +\infty$, $y \to 0$.

- *Turning point(s)*

  $$y' = -\left(x^2 - 1\right)^{-2} \cdot 2x$$

  When $y' = 0$, $x = 0$ (one turning point).

- *Asymptote(s)*

  $x = \pm 1$ and $y = 0$

  When $x = +1^+$, $y \to +\infty$

  When $x = +1^-$, $y \to -\infty$

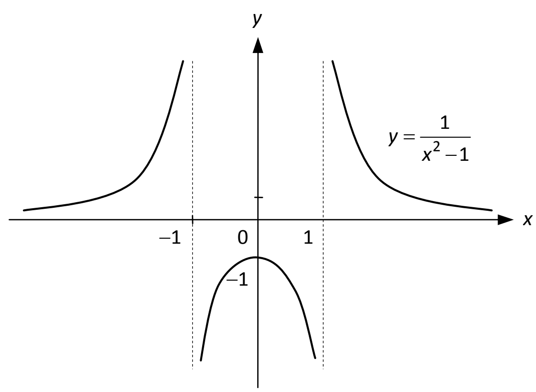
  When $x = -1^+$, $y \to -\infty$

  When $x = +1^-$, $y \to +\infty$

- *Symmetry*

  The curve is symmetrical about y–axis.

- *Sketch*



Please check against the graphic calculator.

(ans) [1]

(ii) **If the domain of f is further restricted to $x \geq k$,**

$\therefore$ **The least value of k for which the function f^{-1} exists**

$k = 0$ **(ans)**

because the function f^{-1} exists if and only if f is a one–one function. **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer.

For the rest of the question, the domain of f is $x \in \mathbb{R}, \quad x \neq \pm 1$ (as originally defined).

The function g is defined as

$$g: \ x \ \mapsto \ \frac{1}{x-3} \quad \text{for} \quad x \in \mathbb{R}, \quad x \neq 2, x \neq 3, x \neq 4$$

(iii) $\therefore$ **To show that** $\mathbf{fg(x) = \dfrac{(x-3)^2}{(4-x)(x-2)}}$ **:**

$$\text{LHS} = \text{fg}(x)$$

$$= \text{f}\left[\text{g}(x)\right] = \text{f}\left[\frac{1}{x-3}\right] = \frac{1}{\left(\dfrac{1}{x-3}\right)^2 - 1}$$

$$= \frac{(x-3)^2}{1-(x-3)^2} = \frac{(x-3)^2}{1-x^2+6x-9} = \frac{(x-3)^2}{-x^2+6x-8}$$

$$= \frac{(x-3)^2}{(-x+4)(x-2)}$$

$$= \frac{(x-3)^2}{(4-x)(x-2)} = \text{RHS} \quad \textbf{(QED)} \quad [2]$$

☺ **Exam Report**

Most candidates gave the correct answer.

(iv) $\therefore$ **To solve the inequality** $\mathbf{fg(x) > 0}$ **:**

$$\text{fg}(x) = \frac{(x-3)^2}{(4-x)(x-2)} > 0$$

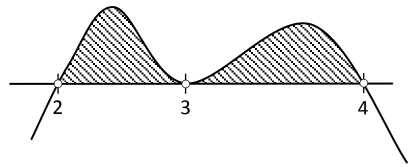
Multiplying both sides by $(4-x)^2(x-2)^2$,

$$\frac{(x-3)^2}{(4-x)(x-2)}(4-x)^2(x-2)^2 > 0 \cdot (4-x)^2(x-2)^2$$

$$(4-x)(x-2)(x-3)^2 > 0$$

Sketching the curve,

Critical values are 2 and 3 and 4.



$\therefore$ **Solution set of the inequality:**

$2 < x < 3$ or $3 < x < 4$ **(ans)** [3]

☺ **CheckBack**

If the answer is $2 < x < 3$ or $3 < x < 4$,

- Test $x = 2½$

$$\text{fg}(x) = \frac{(2\tfrac{1}{2}-3)^2}{(4-2\tfrac{1}{2})(2\tfrac{1}{2}-2)} > 0$$

- Test $x = 3½$

$$\text{fg}(x) = \frac{(3\tfrac{1}{2}-3)^2}{(4-3\tfrac{1}{2})(3\tfrac{1}{2}-2)} > 0$$

(checked)

☺ **Exam Report**

Almost all candidates gave the correct answer.

(v) $\therefore$ Let R_{fg} be the range of fg.

The range of g is fed into the function f.

$R_{\text{g}} = (-\infty, +\infty)$, except $y = 0$

$D_{\text{f}} = R_{\text{g}}$

Since D_{f} is almost full $x \in \mathbb{R}, \quad x \neq 0$,

The range of the function fg is also the range of function f.

$\therefore$ **The range of fg**

$R_{\text{fg}} = (0, \infty) \cup (-\infty, -1)$ **(ans)** [3]

☺ **Exam Report**

Not many candidates gave the correct answer. Even for those near–correct answers, the answer erroneously centered around $(0, \infty) \cup (-\infty, -1]$.

Candidates were advised to understand the correlation between the range and domain of each function in a composite mode.

§

Section B: Statistics [60 marks]

Answer all questions.

5. [Statistics]

Method

Approach I – sampling

Olympics committee plans to hold its international sporting event in city A. To find out more about the opinions of the spectators about its planned catering facilities, a sample of 1% of the spectators is taken.

(i) $\therefore$ **The reason why it would be difficult to use a stratified sample** is

Because it is near impossible to divide the spectator population into **independent homogeneous subgroups** before *sampling*. The data collected of spectator population (such as age, culture, race, religion, habits or group size) at the entrance deem it near impracticable to have it **exhaustively collected** and divided into **mutually exclusive groups**. **(ans)** [1]

☺ **Stratified sampling**

Stratified sampling is *sampling* from the *population* and *stratification* is the process of dividing members of the population into independent homogeneous subgroups before *sampling*.

- The strata should be **mutually exclusive**: every element in the population must be assigned to only one stratum.
- The strata should also be **collectively exhaustive**: no population element can be excluded. Then random or systematic sampling is applied within each stratum. This often improves the representativeness of the sample by reducing sampling error. It can produce a **weighted mean** that has less variability than the **arithmetic mean** of a simple random sample of the population.

(ii) $\therefore$ **A systematic sample could be carried out.**

This is random sampling with a system. From the sampling frame, a starting point is chosen at random, and choices thereafter are at regular intervals. For example, suppose one wants to sample 100 spectators from a population of 10000 spectators. 1% of 10000=100, so every 100th spectator is chosen after a random starting point between 1 and 100. If the random starting point is 11, then the spectator selected are 11, 111, 211, 311, 411, 511, 611, and 71, etc. **(ans)** [2]

☺ **Systematic sampling**

Systematic sampling is a statistical method involving the selection of elements from an **ordered sampling frame**. The most common form of systematic sampling is an **equal-probability method**, in which every k^{th} element in the frame is selected, where k, the **sampling interval** (sometimes known as the **skip**), is calculated as:

$$k = \frac{N}{n}$$

where n is the sample size, and N is the population size.

This is one of the methods used.

Examples

❶ Suppose a supermarket wants to study buying habits of their customers, then using *systematic sampling* they can choose every 10th or 15th customer entering the supermarket and conduct the study on this sample.

This is random sampling with a system. From the sampling frame, a starting point is chosen at random, and choices thereafter are at regular intervals.

❷ For example, suppose you want to sample 8 houses from a street of 120 houses. 120/8=15, so every 15th house is chosen after a random starting point between 1 and 15. If the random starting point is 11, then the houses selected are 11, 26, 41, 56, 71, 86, 101, and 116.

§

6. **[Statistics]**

Method

Approach I – hypothesis testing

There is a claim that providing background music in office environment affects the time required by an employee to complete a task.

For a particular office, without background music, the time required by an employee to complete a task is a normally distributed random variable. Over a long period of survey it is known that the mean time required is 42.0 minutes. Background music is then introduced in the workplace. After a while, a random sample of 11 employees is chosen to measure their time required to complete a task, t minutes. The results are summarized as follows.

$N = 11 \qquad \Sigma t = 454.3 \qquad \Sigma t^2 = 18778.43$

∴ **Unbiased estimate of the population mean**

$$\overline{X} = \frac{\sum x}{n}$$

$$= \frac{\sum t}{N} = \frac{454.3}{11}$$

$= 41.3$ (3sf) **(ans)**

∴ **Unbiased estimate of the population variance**

$$s^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\overline{x}\right)^2\right]$$

$$= \frac{N}{N-1}\left[\frac{\sum t^2}{N} - \left(\overline{X}\right)^2\right]$$

$$= \frac{11}{11-1}\left[\frac{18778.43}{11} - (41.3)^2\right]$$

$= 1.584$

$= 1.58$ (3sf) **(ans)**

☺ **Exam Report**

A number of candidates wrongly placed the number of significant figures as 2, it should be at least 3.

(ii) ∴ **Carrying out a test at the 10% significant level, whether there has been a change in the mean time required by an employee to complete the task.**

A two–tailed t–test is most suitable.

Test $H_0 : \mu = 42.0$ mins against $H_1 : \mu \neq 42.0$ mins

Significance level: $\alpha = 0.05$

Test static: $T = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$

(*For small sample sizes*)

$$T = \frac{41.3 - 42.0}{\sqrt{1.584}/\sqrt{11}} = -1.8446$$

Using GC,

Critical p–value $= -1.812$ ($v = 10$, 5%)

Since $-1.8446 < -1.812$,

Hence, H_0 is rejected. There is sufficient evidence that there has been a change in the mean time required by an employee to complete the task. **(ans)** [7]

☺ **Unbiased estimates of population mean and population variance**

For a *population* with unknown mean μ and unknown variance σ^2, if $\{X_1, X_2, X_3, \ldots, X_n\}$ is a *random sample* of size n drawn from the *population*,

❶ **Sample mean,**

$$\overline{x} = \frac{\sum x}{n}$$

If the sample values are given in the form of their deviation from a fixed value k, the collected data will be $(x_1 - k), (x_2 - k), \ldots, (x_n - k)$.

$$\overline{x} = \frac{1}{n}\sum(x-k) + k$$

❷ **Sample variance,**

$$s^2 = \frac{1}{n}\sum(x-\overline{x})^2 = \frac{1}{n}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$$

If the sample values are given in the form of their deviation from a fixed value k, the corresponding sample variance will be

$$s^2=\frac{1}{n}\left(\sum(x-k)^2-\frac{\left[\sum(x-k)\right]^2}{n}\right)$$

❸ The **unbiased estimate of the population mean**, $\hat{\mu}$, is given by

$$\hat{\mu}=\bar{x}=\frac{1}{n}\sum x \quad \text{or} \quad \frac{1}{n}\sum(x-k)+k$$

❹ The **unbiased estimate of the population variance**, $\hat{\sigma}^2$, is given by

$$\hat{\sigma}^2=\frac{n}{n-1}s^2=\frac{1}{n-1}\left(\sum x^2-\frac{\left(\sum x\right)^2}{n}\right)$$

$$=\frac{1}{n-1}\left(\sum(x-k)^2-\frac{\left[\sum(x-k)\right]^2}{n}\right)$$

☺ **Exam Report**

It was expected that candidates used a greater accuracy such as correct to four significant figures to calculate the final answer. The final answer can then be presented correct to 3 significant figures. The answer mark was not awarded for inaccurate numerical answer.

§

7. [Statistics]

Method

Approach I – set probability

In a Venn diagram, it is given that for events *A* and *B*,

- $P(A)=0.7$
- $P(B)=0.6$
- $P(A|B')=0.8$

(i) $\therefore$ To find $P(A\cap B')$:

$$P(A|B')=\frac{P(A\cap B')}{P(B')} \Rightarrow$$

$$0.8=\frac{P(A\cap B')}{1-P(B)} \Rightarrow 0.8=\frac{P(A\cap B')}{1-0.6} \Rightarrow$$

$$0.8=\frac{P(A\cap B')}{0.4} \Rightarrow$$

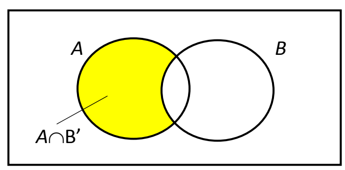
$$P(A\cap B')=0.8\times 0.4$$

$\therefore$ $\mathbf{P(A\cap B')}=0.32$ **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer.

(ii) $\therefore$ To find $P(A\cup B)$:



$$P(A\cap B)=P(A)-P(A\cap B')$$

$$=0.7-0.32=0.38$$

$\therefore$ $\mathbf{P(A\cup B)}=P(A)+P(B)-P(A\cap B)$

$$=0.7+0.6-0.38$$

$$=0.92$$ **(ans)** [2]

(iii) $\therefore$ **P$(B'|A)$**

$$= \frac{P(B' \cap A)}{P(A)}$$

$$= \frac{0.32}{0.7}$$

$= 0.4571$ **(ans)** [2]

> ☺ **Exam Report**
>
> It was expected that candidates used a greater accuracy such as correct to four significant figures to calculate the final answer. The final answer can then be presented correct to 3 significant figures. The answer mark was not awarded for inaccurate numerical or fractional answer such as $\frac{16}{35}$.

For a third event C, it is further given that

- $P(C) = 0.5$
- Events A and C are independent

(iv) $\therefore$ **P$(A' \cap C)$**

$= P(A') \times P(C)$

$= (1 - 0.7) \times 0.5$

$= 0.3 \times 0.5$

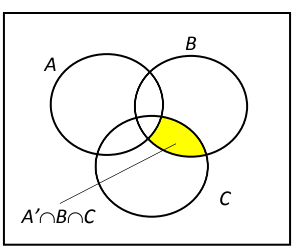
$= 0.15$ **(ans)** [2]

(v) **Hence,**

$\therefore$ **An inequality satisfied by P$(A' \cap B \cap C)$**

$\Rightarrow \; P(A' \cap B \cap C) \le P(A' \cap C)$

$\Rightarrow \; P(A' \cap B \cap C) \le 0.15$ **(ans)** [1]



> ☺ **Exam Report**
>
> Most candidates managed to derive the simple inequality using the hence statement.

§

8. [Statistics]

Method

Approach I – permutations

In a simple number blocks game. Five numbered blocks ❶, ❷, ❸, ❹ and ❺ are arranged randomly to form a five–digit number. There are no other numbered blocks.

(i) $\therefore$ To find the probability that number is greater than 30 000:

Without restrictions, the first numbered block can have all 5 choices, followed by the rest of the numbered blocks.

The choice:

❺❹❸❷❶ $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

With restrictions, the first numbered block can have ❸, ❹ or ❺, followed by the rest of the numbered blocks.

The choice:

❸❹❸❷❶ $= 3 \times 4 \times 3 \times 2 \times 1 = 72$

$\therefore$ **Probability that number is greater than 30 000**

$$= \frac{\text{restricted permutations}}{\text{total permutations}}$$

$$= \frac{72}{120}$$

$= \frac{3}{5}$ **(ans)** [1]

> ☺ **Exam Report**
>
> It was expected for candidates to give exact answer for this question, although the answer, 0.6 was accepted.

(ii) $\therefore$ To find the probability that the last two digits are both even:

With restrictions, the last 2 digits must either be ❷❹ or ❹❷ (2 choices). The first 3 digits are permutated as per normal.

The choice:

❸❷❶❷❶ $= 3 \times 2 \times 1 \times 2 \times 1 = 12$

$\therefore$ **Probability that probability that the last two digits are both even**

$$= \frac{\text{restricted permutations}}{\text{total permutations}}$$

$$= \frac{12}{120}$$

$= \frac{1}{10}$ **(ans)** [2]

☺ **Exam Report**

It was expected for candidates to give exact answer for this question, although the answer, 0.1 was accepted.

(iii) $\therefore$ To find the probability that the number is greater than 30 000 and odd:

- *Case 1*:

  The first digit is ❸. (1 choice)

  The last digit can either be ❶ or ❺. (2 choices)

  The rest of the 3 digits are permutated arranged as per normal.

  The choice:

  ❶❸❷❶❷ $= 1 \times 3 \times 2 \times 1 \times 2 = 12$

- *Case 2*:

  The first digit is ❹. (1 choice)

  The last digit can either be ❶ or ❸ or ❺. (3 choices)

  The rest of the 3 digits are permutated arranged as per normal.

  The choice:

  ❶❸❷❶❸ $= 1 \times 3 \times 2 \times 1 \times 3 = 18$

- *Case 3*:

  The first digit is ❺. (1 choice)

  The last digit can either be ❶ or ❸. (2 choices)

  The rest of the 3 digits are permutated arranged as per normal.

  The choice:

  ❶❸❷❶❷ $= 1 \times 3 \times 2 \times 1 \times 2 = 12$

$\therefore$ **Probability that the number is greater than 30 000 and odd**

$$= \frac{\text{restricted permutations}}{\text{total permutations}}$$

$$= \frac{12 + 18 + 12}{120} = \frac{42}{120}$$

$= \frac{7}{20}$ **(ans)** [4]

☺ **Exam Report**

It was expected for candidates to give exact answer for this question, although the answer, 0.35 was accepted.

§

9. [Statistics]

Method

Approach I – normal distribution

[In this question, give answer by stating clearly the values of the parameters of any normal distribution used.]

The telephone company charges different rates for telephone calls made during the peak and off–peak periods. Over a three–month period, Keith makes X minutes of peak–rate calls and Y minutes of off–peak–rate calls.

It is given that X and Y are independent random variables with distributions $\mathrm{N}(180, 30^2)$ and $\mathrm{N}(400, 60^2)$ respectively.

(i) $\therefore$ Let $\mathrm{P}(Y > 2X)$ be the probability that, over a random three–month period, the number of minutes of off–peak–rate calls made by Keith is more than twice the number of minutes of peak–rate calls. — ❶

From the statements,

$$X \sim \mathrm{N}(180, 30^2)$$

$$Y \sim \mathrm{N}(400, 60^2)$$

From ❶,

$$\mathrm{P}(Y > 2X) = \mathrm{P}(Y - 2X > 0) \quad \text{— ❷}$$

Let $A = Y - 2X$,

$$\mu_A = \mu_Y - 2\mu_X$$

$$= 400 - 2(180) = 40$$

$$\sigma_A{}^2 = \sigma_Y{}^2 + 2^2\sigma_X{}^2$$

$$= 60^2 + 2^2 \cdot 30^2 = 7200$$

$$A \sim \text{N}\left(40, \sqrt{7200}^2\right)$$

From ❷,

$$\text{P}(Y - 2X > 0) = \text{P}(A > 0) \qquad \text{— ❸}$$

Normalizing ❸,

$$\text{P}(A > 0) = \text{P}\left(Z \geq \frac{0 - \mu_A}{\sigma_A}\right)$$

$$= \text{P}\left(Z \geq \frac{0 - 40}{\sqrt{7200}}\right) = \text{P}(Z \geq -0.4714)$$

$$= \text{P}(Z \leq 0.4714) = 0.6813$$

∴ The probability that, over a random three–month period, the number of minutes of off–peak–rate calls made by Keith is more than twice the number of minutes of peak–rate calls

$\text{P}(Y > 2X) = 0.6813$ (4dp) **(ans)** [4]

☺ **Exam Report**

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

It is further given that peak–rate calls $0.12 per minute and off–peak–rate calls cost $0.05 per minute.

(ii) ∴ Let $\text{P}(\$0.05 \cdot Y + \$0.12 \cdot X > \$45)$ be the probability that, over a random three–month period, the total cost of Ken's calls is greater than $45. — ❹

Let $B = \$0.05 \cdot Y + \$0.12 \cdot X$,

$$\mu_B = \$0.05 \cdot \mu_Y + \$0.12 \cdot \mu_X$$

$$= \$0.05 \cdot (400) + \$0.12 \cdot (180) = \$41.60$$

$$\sigma_B{}^2 = 0.05^2 \cdot \sigma_Y{}^2 + 0.12^2 \cdot \sigma_X{}^2$$

$$= 0.05^2 \cdot 60^2 + 0.12^2 \cdot 30^2 = \$^2\,21.96$$

$$B \sim \text{N}\left(41.60, \sqrt{21.96}^2\right)$$

From ❹,

$$\text{P}(\$0.05 \cdot Y + \$0.12 \cdot X > \$45) = \text{P}(B > \$45) \qquad \text{— ❺}$$

Normalizing ❺,

$$\text{P}(B > \$45) = \text{P}\left(Z \geq \frac{45 - \mu_B}{\sigma_B}\right)$$

$$= \text{P}\left(Z \geq \frac{45 - 41.6}{\sqrt{21.96}}\right) = \text{P}(Z \geq 0.7255)$$

$$= 1 - \text{P}(Z \leq 0.7255) = 1 - 0.76585$$

$$= 0.2342$$

∴ The probability that, over a random three–month period, the total cost of Ken's calls is greater than $45

$\text{P}(\$0.05 \cdot Y + \$0.12 \cdot X > \$45)$

$= 0.2342$ (4dp) **(ans)** [3]

☺ **Exam Report**

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

(iii) ∴ Let $\text{P}(\$0.12 \cdot X_1 + \$0.12 \cdot X_2 > \$45)$ be the probability that the total cost of Ken's peak–rate calls over two independent random three–month periods is greater than $45. — ❻

Let $C = \$0.12 \cdot X_1 + \$0.12 \cdot X_2$,

$$\mu_C = \$0.12 \cdot \mu_X + \$0.12 \cdot \mu_X$$

$$= \$0.12(180) + \$0.12(180) = \$43.20$$

$$\sigma_C{}^2 = 0.12^2 \cdot \sigma_X{}^2 + 0.12^2 \cdot \sigma_X{}^2$$

$$= 0.12^2 \cdot 30^2 + 0.12^2 \cdot 30^2 = \$^2\,25.92$$

$$C \sim \text{N}\left(43.20, \sqrt{25.92}^2\right)$$

From ❻,

$$P\left(\$0.12 \cdot X_1 + \$0.12 \cdot X_2 > \$45\right) = P(C > \$45)$$

— ❼

Normalizing ❼,

$$P(C > \$45) = P\left(Z \geq \frac{45 - \mu_C}{\sigma_C}\right)$$

$$= P\left(Z \geq \frac{45 - 43.2}{\sqrt{25.92}}\right) = P(Z \geq 0.35355)$$

$$= 1 - P(Z \leq 0.35355) = 1 - 0.6381$$

$$= 0.3619$$

∴ **The probability that the total cost of Ken's peak–rate calls over two independent random three–month periods is greater than $45**

$$P\left(\$0.12 \cdot X_1 + \$0.12 \cdot X_2 > \$45\right)$$

$= 0.3619$ (4dp) **(ans)** [3]

☺ **Exam Report**

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

§

10. [Statistics]

Method

Approach I – product moment correlation coefficient

The wind resistance of a car is being tested in a wind tunnel. The drag force *F* for different wind speeds *v*, in appropriate units, is measured and recorded. The results are shown in the table below.

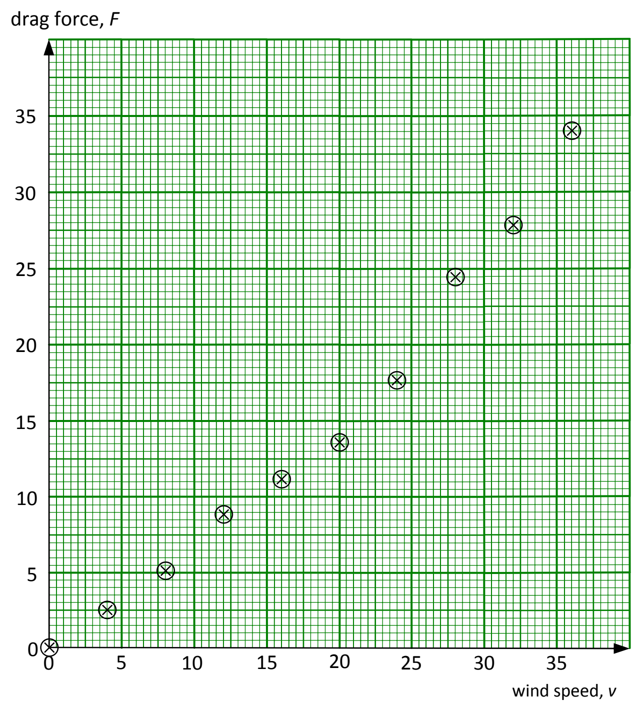
v	0	4	8	12	16	20	24	28	32	36
F	0	2.5	5.1	8.8	11.2	13.6	17.6	22.0	27.8	33.9

(i) **Labelling the axes clearly.**

∴ **A scatter diagram for these values** is drawn.

Independent variable is the wind speed, *v*.

Dependent variable is the drag force, *F*.



Scatter Diagram

(ans) [2]

Based on the theoretical analysis, it is suggested that the drag force *F* may be modelled by one of the formulae

$$F = a + bv \quad \text{or} \quad F = c + dv^2$$

where *a*, *b*, *c* and *d* are constants.

(ii) [For this question, give answers correct to 4 decimal places.]

(a) $\therefore$ Let r_1 be the value of the product moment correlation coefficient between *v* and *F*.

From the table, the following values are computed.

v	0	4	8	12	16	20	24	28	32	36
F	0	2.5	5.1	8.8	11.2	13.6	17.6	22.0	27.8	33.9

$n = 10$

$\sum_{i=1}^{10} v_i = 180 \qquad \sum_{i=1}^{10} v_i^2 = 4560 \qquad \bar{v} = 18.0$

$\sum_{i=1}^{10} F_i = 142.5 \qquad \sum_{i=1}^{10} F_i^2 = 3135.91 \qquad \bar{F} = 14.25$

$\sum_{i=1}^{10} v_i \cdot F_i = 3756$

$$S_{vv} = \sum_{i=1}^{10} v_i^2 - \frac{\left(\sum_{i=1}^{10} v_i\right)^2}{10}$$

$$= 4560 - \frac{180^2}{10} = 1320 \; = 1320$$

$$S_{vF} = \sum_{i=1}^{10} v_i \cdot F_i - \frac{\left(\sum_{i=1}^{10} v_i\right)\left(\sum_{i=1}^{10} F_i\right)}{10}$$

$$= 3756 - \frac{180 \times 142.5}{10} = 1191$$

$$S_{FF} = \sum_{i=1}^{10} F_i^2 - \frac{\left(\sum_{i=1}^{10} F_i\right)^2}{10}$$

$$= 3135.91 - \frac{142.5^2}{10} = 1105.285$$

$\therefore$ **The value of the product moment correlation coefficient between *v* and *F***

$$r_1 = \frac{S_{vF}}{(S_{vv} \cdot S_{FF})^{\frac{1}{2}}}$$

$$= \frac{1191}{(1320 \times 1105.285)^{\frac{1}{2}}}$$

$$= 0.98602439$$

$= 0.9860$ (4dp) **(ans)**

☺ **Exam Report**

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

Some candidates still did not know how to use the graphic calculator correctly and hence, lead to wrong answer such as wrong values inputed.

(b) $\therefore$ Let r_2 be the value of the product moment correlation coefficient between v^2 and *F*.

From the table, the following values are computed.

$v^2 \equiv X$	0	16	64	144	256	400	576	784	1024	1296
F	0	2.5	5.1	8.8	11.2	13.6	17.6	22.0	27.8	33.9

$n = 10$

$\sum_{i=1}^{10} X_i = 4560 \qquad \sum_{i=1}^{10} X_i^2 = 3925248 \qquad \bar{X} = 456.0$

$\sum_{i=1}^{10} F_i = 142.5 \qquad \sum_{i=1}^{10} F_i^2 = 3135.91 \qquad \bar{F} = 14.25$

$\sum_{i=1}^{10} X_i \cdot F_i = 109728$

$$S_{XX} = \sum_{i=1}^{10} X_i^2 - \frac{\left(\sum_{i=1}^{10} X_i\right)^2}{10}$$

$$= 3925248 - \frac{4560^2}{10} = 1845888$$

$$S_{XF} = \sum_{i=1}^{10} X_i \cdot F_i - \frac{\left(\sum_{i=1}^{10} X_i\right)\left(\sum_{i=1}^{10} F_i\right)}{10}$$

$$= 109728 - \frac{4560 \times 142.5}{10} = 44748$$

$$S_{FF} = \sum_{i=1}^{10} F_i^2 - \frac{\left(\sum_{i=1}^{10} F_i\right)^2}{10}$$

$$= 3135.91 - \frac{142.5^2}{10} = 1105.285 \quad \text{(as before)}$$

$\therefore$ **The value of the product moment correlation coefficient between v^2 and *F***

$$r_2 = \frac{S_{XF}}{(S_{XX} \cdot S_{FF})^{\frac{1}{2}}}$$

$$= \frac{44748}{(1845888 \times 1105.285)^{\frac{1}{2}}}$$

$$= 0.99068$$

$= 0.9907$ (4dp) **(ans)**

☺ **Exam Report**

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

Some candidates still did not know how to use the graphic calculator correctly and hence, lead to wrong answer such as wrong values inputed.

(iii) **Use your answer to parts (i) and (ii),**

$\therefore$ The *product moment correlation coefficient* ranges from -1 to 1. A value of 1 implies that a linear equation describes the relationship between the independent variable, X and dependent variable, Y perfectly, with all data points lying on a line for which Y increases as X increases. A value of -1 implies that all data points lie on a line for which Y decreases as X increases. A value of 0 implies that there is no linear correlation between the variables.

Since $\boldsymbol{F = c + dv^2}$ has a *product moment correlation coefficient* closer to 1 than $\boldsymbol{F = a + bv}$, it implies that $F = c + dv^2$ is a slightly **better model**. **(ans)** [1]

(iv) $\therefore$ To find the equation of a suitable regression line:

Better model: $F = c + dv^2$

Thus,

$$d = \frac{S_{XF}}{S_{XX}}$$

$$= \frac{44748}{1845888} = 0.02424199$$

$$c = \overline{F} - d\overline{X}$$

$$= 14.25 - 0.02424199 \times 456.0$$

$$= 3.195652173913$$

$\therefore$ **The equation of a suitable regression line**

$$F = 3.195652173913 + 0.02424199v^2 \text{ — ❶}$$

$F \approx 3.1957 + 0.0242v^2$ (4dp) **(ans)**

Using the suitable regression line,

$\therefore$ Let v_R be the required estimate value of v for which $F = 26.0$.

From ❶,

$$F = 3.195652173913 + 0.02424199v^2$$

When $F = 26.0$,

$$26.0 = 3.195652173913 + 0.02424199v^2$$

$$26.0 - 3.195652173913 = 0.02424199v^2$$

$$22.804347826 = 0.02424199v^2$$

$$\frac{22.804347826}{0.02424199} = v^2$$

$\Rightarrow$ $v = 30.67$ (4sf) **(ans)**

$\therefore$ **The reason why neither the regression line of v and F nor the regression line of v^2 and F should be used** is

Since F is a dependent variable, and not an independent variable, one should not use the regression line of v on F or the regression line of v^2 on F to estimate v. **(ans)** [4]

☺ **Exam Report**

It was expected for candidates to give answer up to 4 significant figures. The final answer mark was not awarded for less accurate answer.

Many reasons suggested by the candidates were deemed not suitable for the answer mark to be awarded.

§

11. [Statistics]

Method

Approach I – Poisson distribution

[In this question, give answer by stating clearly all distributions used, together with values of the appropriate parameters.]

The number of calls received by a call centre is being recorded for a long time. It is found that the number of telephone calls received by a call centre in one minute is a random variable with Po(3).

(i) $\therefore$ Let $\mathrm{P}\left(X_{4\text{ mins}}=8\right)$ be the probability that exactly 8 calls are received in a randomly chosen period of 4 minutes. — ❶

Let X be the random variable denoting the number of telephone calls received by a call centre in one minute.

$$\Rightarrow \quad X \sim \mathrm{Po}\left(\lambda_X=3\right)$$

Let A be the random variable denoting the number of telephone calls received by a call centre in 4 minutes.

$$\Rightarrow \quad A \sim \mathrm{Po}\left(\lambda_A=4\cdot 3\right)=\mathrm{Po}\left(\lambda_A=12\right)$$

$\therefore$ **The probability that exactly 8 calls are received in a randomly chosen period of 4 minutes**

$$\mathrm{P}\left(X_{4\text{ mins}}=8\right) = \mathrm{P}\left(A=8\right)$$

$$=\frac{{\lambda_A}^{8}\cdot \mathrm{e}^{-\lambda_A}}{8!}=\frac{12^{8}\cdot \mathrm{e}^{-12}}{8!}$$

$=0.06552 \approx 0.0655$ (4dp) **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer.

It was expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

(ii) $\therefore$ Let T be the length of time, to the nearest second, for which probability that no calls are received is 0.2.

Let B be the random variable denoting the number of telephone calls received by a call centre in 1 second.

$$\Rightarrow \quad B \sim \mathrm{Po}\left(\lambda_B=\tfrac{3}{60}\right)$$

$\therefore$ Probability that no calls are received in a randomly chosen period of 1 second

$$=\frac{{\lambda_B}^{0}\cdot \mathrm{e}^{-\lambda_B}}{0!}$$

$$=\frac{\left(\frac{3}{60}\right)^{0}\cdot \mathrm{e}^{-\frac{3}{60}}}{0!}=0.9512294245$$

For each successive no–call second, the probability is multiplied together,

$$\Rightarrow \quad 0.9512294245^{T}=0.2$$

$$\Rightarrow \quad T\log\left(0.9512294245\right)=\log\left(0.2\right)$$

$$\Rightarrow \quad T=\frac{\log\left(0.2\right)}{\log\left(0.9512294245\right)}=32.18 \text{ seconds}$$

$\therefore$ **The length of time, to the nearest second, for which probability that no calls are received is 0.2**

$T=32$ seconds (to the nearest second) **(ans)** [3]

☺ **Exam Report**

More than half of the candidates gave the incorrect answer.

It was also expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

(iii) **Use a suitable approximation,**

$\therefore$ Let $\mathrm{P}\left(X_{12\text{ hours}}>2200\right)$ be the probability that, on a randomly chosen working day of 12 hours, more than 2200 calls are received.

Let C be the random variable denoting the number of telephone calls received by a call centre in a randomly chosen working day of 12 hours.

$$\Rightarrow \quad C \sim \mathrm{Po}\left(\lambda_C=3\times 60\times 12=2160\right)$$

Given $X \sim \mathrm{Po}\left(\lambda\right)$ (*Poisson distribution*) where $\mathrm{E}\left(X\right)=\lambda$ and $\mathrm{Var}\left(X\right)=\lambda$, if $\lambda>10$, then X can be approximated by a *normal distribution*.

$$\boxed{X \sim \mathrm{Po}\left(\lambda\right)\approx \mathrm{N}\left(\lambda,\lambda\right)}$$

$$\Rightarrow \quad C \approx \mathrm{N}\left(2160, \sqrt{2160}^2\right)$$

∴ The probability that, on a randomly chosen working day of 12 hours, more than 2200 calls are received

$$\mathrm{P}(X_{12\text{ hours}} > 2200) = \mathrm{P}(C > 2200) \text{ — ❷}$$

Normalizing ❷,

$$\mathrm{P}(C > 2200)_{\text{discrete}}$$

$$= \mathrm{P}(C > 2200 + 0.5)_{\text{continous}}$$

$$= \mathrm{P}\left(Z \geq \frac{2200.5 - \mu_C}{\sigma_C}\right)$$

$$= \mathrm{P}\left(Z \geq \frac{2200.5 - 2160}{\sqrt{2160}}\right)$$

$$= \mathrm{P}(Z \geq 0.87142)$$

$$= 1 - \mathrm{P}(Z \leq 0.87142) = 1 - 0.8082$$

$= 0.1918$ (4dp) **(ans)** [4]

☺ **Normal approximation to Poisson distribution**

- Given $X \sim \mathrm{Po}(\lambda)$ (*Poisson distribution*) where $\mathrm{E}(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$, if $\lambda > 10$, then X can be approximated by a *normal distribution*.

$$\boxed{X \sim \mathrm{Po}(\lambda) \approx \mathrm{N}(\lambda, \lambda)}$$

- **Continuity correction** is to be applied when a *Poisson distribution* (discrete) is approximated by a *normal distribution* (continuous).

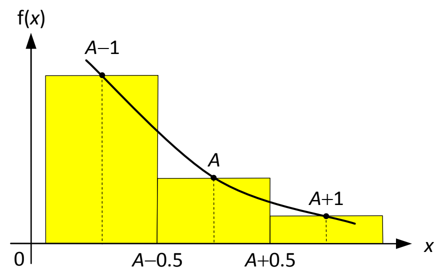
☺ **Continuity correction**

A process known as **continuity correction** needs to be applied when a *binomial distribution* (discrete) is approximated by a *normal distribution* (continuous).

The rules below are used in *continuity corrections*.

Given A is an integer,

$$\mathrm{P}_{\text{discrete}}(X = A) \rightarrow \mathrm{P}_{\text{continuous}}(A - 0.5 < X < A + 0.5)$$
$$\mathrm{P}_{\text{discrete}}(X \geq A) \rightarrow \mathrm{P}_{\text{continuous}}(X > A - 0.5)$$
$$\mathrm{P}_{\text{discrete}}(X > A) \rightarrow \mathrm{P}_{\text{continuous}}(X > A + 0.5)$$
$$\mathrm{P}_{\text{discrete}}(X \leq A) \rightarrow \mathrm{P}_{\text{continuous}}(X < A + 0.5)$$
$$\mathrm{P}_{\text{discrete}}(X < A) \rightarrow \mathrm{P}_{\text{continuous}}(X < A - 0.5)$$



☺ **Exam Report**

Most candidates gave the correct answer.

It was also expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

A "busy" day (S) is a working day of 12 hours on which more than 2200 calls are received.

(iv) ∴ The probability that, in six randomly chosen working days, exactly two are busy

$$S \sim \mathrm{B}(6, 0.1918)$$

$$\mathrm{P}(S = 2)$$

$$= {}^6\mathrm{C}_2 (0.1918)^2 (1 - 0.1918)^4$$

$$= 15(0.1918)^2(0.8082)^4$$

$$= 0.2354311$$

$= 0.2354$ (4dp) **(ans)** [2]

☺ **Exam Report**

Most candidates gave the correct answer.

It was also expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

(v) Use a suitable approximation,

$\therefore$ Let $\mathrm{P}(S<10)$ be the probability that, in 30 randomly chosen working days of 12 hours, fewer than 10 are busy.

$S \sim \mathrm{B}(30, 0.1918)$

Given that $n=30$ and $p=0.1918$.

Since $n=30$ (large),

$np=30(0.1918)=5.754>5$ and

$n(1-p)=30(0.8082)=24.246>5$,

The binomial distribution can be **approximated** to the normal distribution with mean $np=5.754$ and variance $np(1-p)=4.6504$.

$\Rightarrow \quad S \approx \mathrm{N}\left(5.754, \sqrt{4.6504}^2\right)$

$\therefore$ **The probability that, in 30 randomly chosen working days of 12 hours, fewer than 10 are busy**

$\mathrm{P}(S<10)_{\text{discrete}} = \mathrm{P}(S<9.5)_{\text{continous}}$ — ❸

Normalizing ❸,

$\mathrm{P}(S<9.5)_{\text{continous}}$

$= \mathrm{P}\left(Z \le \frac{9.5-\mu_S}{\sigma_S}\right) = \mathrm{P}\left(Z \le \frac{9.5-5.754}{\sqrt{4.6504}}\right)$

$= \mathrm{P}(Z \le 1.7371)$

$= 0.9588$ (4dp) **(ans)** [4]

☺ **Normal approximation to binomial distribution**

- Given $X \sim \mathrm{B}(n,p)$ (*binomial distribution*) where $\mathrm{E}(X)=np$ and $\mathrm{Var}(X)=npq$, if n is large $(n>50)$ and $np>5$ $\boxed{\text{and } nq>5}$, then X can be approximated by a *normal distribution*.

  $X \sim \mathrm{B}(n,p) \approx \mathrm{N}(np, npq)$

- This *approximation* gets better as $n \to \infty$ and/or $p \to 0.5$.

☺ **Exam Report**

Most candidates gave the correct answer.

It was also expected for candidates to give answer up to 4 decimal places. The final answer mark was not awarded for less accurate answer.

§

Notes: