2011

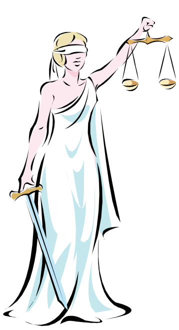
nov

advanced

mathematics

complete yearly solutions

9740



2011 Nov Paper 1 9740/1

Paper 1 (3 hours) consisting of about 10 to 12 questions of different lengths and marks based on the Pure Mathematics section of the syllabus.

*Candidates will be expected to answer **all** questions.*

Questions

Answer all questions.

1. [Functions & graphs]

Solution

[This question is to be solved without using a calculator.]

$\therefore$ **The inequality,**

$$\frac{x^2+x+1}{x^2+x-2}<0 \quad ❶$$

$$\Rightarrow \frac{x^2+x+1}{(x+2)(x-1)}<0 \quad \text{(factorise the denominator)}$$

$$\Rightarrow \frac{x^2+x+1}{(x+2)(x-1)}\cdot(x+2)^2(x-1)^2<0\cdot(x+2)^2(x-1)^2$$

$$\Rightarrow \left(x^2+x+1\right)(x+2)(x-1)<0$$

$$\Rightarrow \left(x^2+x+\tfrac{1}{4}+\tfrac{3}{4}\right)(x+2)(x-1)<0$$

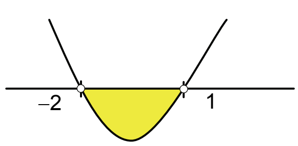
$$\Rightarrow \left[\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right](x+2)(x-1)<0 \quad ❷$$

Regardless of any real values of *x*, the expression, $\left[\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right]$ of ❷ will remain positive, hence, it can be divided away without affecting the inequality.

$$❷ \Rightarrow \frac{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right](x+2)(x-1)}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]}<\frac{0}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]}$$

$$\Rightarrow (x+2)(x-1)<0$$

Sketching the graph with roots of the equation $x=-2$ and $x=1$:



From above, $-2<x<1$.

$\therefore$ **The set of values of *x* for which** $\frac{x^2+x+1}{x^2+x-2}<0$ **:**

$\{x:\ -2<x<1,\ \ x\in\mathbb{R}\}$ **(ans)** [4]

☺ **CheckBack**

There is no easy CheckBack option for this question.

If the answer is $\{x:\ -2<x<1,\ \ x\in\mathbb{R}\}$,

$\therefore$ Let $x=0$,

LHS of ❶ $=\frac{0^2+0+1}{0^2+0-2}=-½\ <\ 0=$ RHS of ❶

(checked)

- **Exam Report**

  At this level of examination, candidates should be versed in complex numbers too. Hence, the level of answer must commensurate this level of complexity. Some candidates suggested the plain answer as $-2<x<1$ .The answer mark was not credited.

☺ **Checking the Answer**

- Wherever possible do check the answer by substituting a simple numerical solution to the original equation and test its validity.

2. [Functions & graphs]

Solution

[For this question, give answers correct to 3 decimal places.]

Given that $\text{f}(x)=ax^2+bx+c$, where *a*, *b* and *c* are constants.

(i) It is given that the curve with equation $y=\text{f}(x)$ passes through the points with coordinates $(-1.5,4.5)$, $(2.1,3.2)$ and $(3.4,4.1)$.

$\therefore$ **To find the values of *a*, *b* and *c*:**

Applying the boundary conditions:

- $a(-1.5)^2+b(-1.5)+c=4.5$

  $\Rightarrow 2.25a-1.5b+c=4.5$ —— ❶

- $a(2.1)^2 + b(2.1) + c = 3.2$

  $\Rightarrow \quad 4.41a + 2.1b + c = 3.2 \qquad$ — ❷

- $a(3.4)^2 + b(3.4) + c = 4.1$

  $\Rightarrow \quad 11.56a + 3.4b + c = 4.1 \qquad$ — ❸

EITHER

From graphic calculator (GC),

The *augmented matrix* is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{array}{ccc|c} 2.25 & -1.5 & 1 & 4.5 \\ 4.41 & 2.1 & 1 & 3.2 \\ 11.56 & 3.4 & 1 & 4.1 \end{array}\right) = \begin{pmatrix} 0.215 \\ -0.490 \\ 3.281 \end{pmatrix}$$

$a = 0.215$, $b = -0.490$, $c = 3.281$

OR

Using *Gaussian elimination*,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{array}{ccc|c} 2.25 & -1.5 & 1 & 4.5 \\ 4.41 & 2.1 & 1 & 3.2 \\ 11.56 & 3.4 & 1 & 4.1 \end{array}\right)$$

- Row ③ = $\dfrac{11.56}{2.25}$ ×Row ① – Row ③ (making a triangular matrix)

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{array}{ccc|c} 2.25 & -1.5 & 1 & 4.5 \\ 4.41 & 2.1 & 1 & 3.2 \\ 0 & -11.10667 & 4.13778 & 19.02 \end{array}\right)$$

- Row ② = $\dfrac{4.41}{2.25}$ ×Row ① – Row ②

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{array}{ccc|c} 2.25 & -1.5 & 1 & 4.5 \\ 0 & -5.04 & 0.96 & 5.62 \\ 0 & -11.10667 & 4.13778 & 19.02 \end{array}\right)$$

- Row ③ = $\dfrac{11.10667}{5.04}$ ×Row ② – Row ③

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{array}{ccc|c} 2.25 & -1.5 & 1 & 4.5 \\ 0 & -5.04 & 0.96 & 5.62 \\ 0 & 0 & -2.02222 & -6.63518 \end{array}\right)$$

- Evaluating the elements

  $c = 3.281$

  $-5.04b + 0.96c = 5.62 \quad \Rightarrow \quad b = -0.490$

  $2.25a - 1.5b + c = 4.5 \quad \Rightarrow \quad a = 0.215$

OR

Solving them simultaneously by systemically removing their variables

- ❸′ = ❶ – ❸ (no difference from the Gaussian elimination method)

  $-9.31a - 4.9b = 0.4 \qquad$ — ❸′

- ❷′ = ❶ – ❷

  $-2.16a - 3.6b = 1.3 \qquad$ — ❷′

- ❸″ = 2.16×❸′ – 9.31×❷′

  $-10.584b + 33.516b = 0.864 - 12.103$

  $22.932b = -11.239$

  $\Rightarrow \quad b = -0.4901 = -0.490$ (3dp) **(ans)**

- ❷′

  $-2.16a - 3.6(-0.4901) = 1.3$

  $\Rightarrow \quad a = 0.21498 = 0.215$ (3dp) **(ans)**

- ❶

  $2.25(0.21498) - 1.5(-0.4901) + c = 4.5$

  $2.25(0.21498) - 1.5(-0.4901) + c = 4.5$

  $\Rightarrow \quad c = 3.2811 = 3.281$ (3dp) **(ans)**

∴ **The values of *a*, *b* and *c*** are

0.215, –0.490 and 3.281 respectively (3dp). **(ans)** [3]

- **3–variables equations**

  One can solve it by

  - Using a **graphic calculator** (GC).
  - **Gaussian elimination** is an algorithm for solving systems of linear equations, finding the rank of a matrix, and calculating the inverse of an invertible square matrix. *Gaussian elimination* is named after German mathematician and scientist *Carl Friedrich Gauss*.
  - **Simultaneously** by eliminating one variable and then another.

☺ **CheckBack**

There is no easy CheckBack option for this question.

If the answer is 0.215, –0.490, 3.281,

$\therefore$ ❶: $2.25a - 1.5b + c = 4.5$

LHS of ❶ $= 2.25(0.215) - 1.5(-0.490) + 3.281$

$= 4.49975 \;= 4.5$

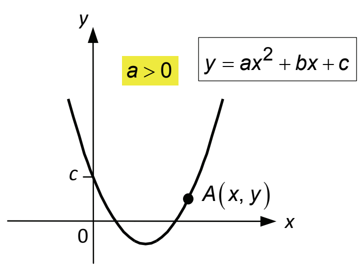
$=$ RHS of ❶ **(checked)**

- **Exam Report**

The candidates were instructed at the beginning of the paper that "unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise."

If there was ambiguity, it was advised that candidates put in their workings in the answer scribe.

(ii) $\therefore$ **To find the set of values of *x* for which f(*x*) is an increasing function**:



Deduce that the function is a quadratic expression and is symmetrical vertically about its minimum point. Any *x*–values to the right of the minimum point has an increasing *y*–value.

$$f(x) = 0.215x^2 - 0.490x + 3.281$$

$$= 0.215\left(x^2 - 2.2790x + 15.260\right)$$

$$= 0.215\left(x^2 - 2.2790x + 1.2984 + 13.961\right)$$

$$= 0.215\left[(x - 1.1395)^2 + 13.961\right]$$

When $x = 1.1395$, $f(x)$ is at its minimum value.

$\therefore$ **The set of values of *x* for which f(*x*) is an increasing function** is

$\{x:\; x > 1.14,\;\; x \in \mathbb{R}\}$ (3sf) **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question.

(checked)

- **Exam Report**

Quite a number of candidates suggested the answer to be $\{x \in \mathbb{R}:\;\; x > 1.14\}$. This was credited as it was in proper set notation. But, candidates were advised to use the more modern form: $\{x:\;\; x > 1.14,\;\; x \in \mathbb{R}\}$.

3. [Functions & graphs]

Solution

Given that the parametric equations of a curve are

$$x = t^2\,,\quad y = \frac{2}{t}$$

(i) [Simplify your answer.]

$\therefore$ To find the equation of the tangent to the curve at the point $\left(p^2, \frac{2}{p}\right)$:

Differentiating the parametric equations *wrt* t:

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = -\frac{2}{t^2}$$

Gradient at point $\left(p^2, \frac{2}{p}\right)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$$

$$\left.\frac{dy}{dx}\right|_p = -\frac{2}{p^2} \bigg/ 2p \;= -\frac{1}{p^3}$$

$\therefore$ **The equation of the tangent to the curve at the point** $\left(p^2, \frac{2}{p}\right)$:

$$\frac{dy}{dx} = \frac{y - y_0}{x - x_0}$$ — in its usual notations

$$-\frac{1}{p^3}=\frac{y-\frac{2}{p}}{x-p^2} \quad\Rightarrow\quad -\frac{1}{p^3}\left(x-p^2\right)=y-\frac{2}{p}$$

$$\Rightarrow\quad -\frac{1}{p^3}x+\frac{1}{p}=y-\frac{2}{p}$$

$$\Rightarrow\quad y=-\frac{1}{p^3}x+\frac{3}{p} \quad \textbf{(ans)} \quad [2] \qquad ❶$$

☺ **CheckBack**

There is no easy CheckBack option for this question. Students may consider backward substitution.

(checked)

(ii) $\therefore$ To find the coordinates of the points Q and R where this tangent meets the x– and y– axes respectively:

Considering ❶,

When $y=0$, $\quad 0=-\frac{1}{p^3}x+\frac{3}{p} \quad\Rightarrow\quad \frac{1}{p^3}x=\frac{3}{p}$

$$\Rightarrow\quad x=3p^2$$

$\therefore$ **The coordinates of the points Q where this tangent meets the y– axes** is

$$\left(3p^2,0\right) \quad \textbf{(ans)}$$

When $x=0$, $\quad y=-\frac{1}{p^3}(0)+\frac{3}{p} \;=\frac{3}{p}$

$\therefore$ **The coordinates of the points R where this tangent meets the y– axes** is

$$\left(0,\frac{3}{p}\right) \quad \textbf{(ans)}$$

(iii) $\therefore$ The mid–point of QR $=\left(\frac{3p^2}{2},\frac{3}{2p}\right)$

As p varies,

$$x=\frac{3p^2}{2} \quad\Rightarrow\quad p=\pm\sqrt{\tfrac{2}{3}x} \qquad ❷$$

$$y=\frac{3}{2p} \quad\Rightarrow\quad p=\frac{3}{2y} \qquad ❸$$

Combining ❷ and ❸,

$$p=\pm\sqrt{\tfrac{2}{3}x}=\frac{3}{2y} \quad\Rightarrow\quad \tfrac{2}{3}x=\left(\frac{3}{2y}\right)^2$$

$$\Rightarrow\quad 8xy^2=27$$

$\therefore$ **The cartesian equation of the locus of the mid–point of QR as p varies** is

$$8xy^2=27 \quad \textbf{(ans)} \quad [3]$$

☺ **CheckBack**

There is no easy CheckBack option for this question. Students may consider backward substitution.

(checked)

• **Exam Report**

A number of candidates did not evaluate the mid–point of the line QR before they proceed to find the locus, hence, led to the wrong expression, *i.e.*, $xy^2=27$.

4. [Calculus]

Solution

(i) $\therefore$ **Using the first three non–zero terms of the Maclaurin series for cos x,**

$$\cos x\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

To find the Maclaurin series for g(x), where g(x) = $\cos^6 x$, up to and including the term in x^4:

$$g(x)=\cos^6 x$$

$$=\left(\cos x\right)^6 \;=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)^6$$

$$=\left[1+x^2\left(-\tfrac{1}{2}+\tfrac{1}{24}x^2\right)\right]^6$$

$$=1+\binom{6}{1}x^2\left(-\tfrac{1}{2}+\tfrac{1}{24}x^2\right)+\binom{6}{2}x^4\left(-\tfrac{1}{2}+\tfrac{1}{24}x^2\right)^2+\ldots$$

$$=1+6x^2\left(-\tfrac{1}{2}+\tfrac{1}{24}x^2\right)+15x^4\left(-\tfrac{1}{2}+\tfrac{1}{24}x^2\right)^2+\ldots$$

$$=1-3x^2+\tfrac{1}{4}x^4+15x^4\left(\tfrac{1}{4}+\ldots\right)+\ldots$$

$$=1-3x^2+\tfrac{1}{4}x^4+\tfrac{15}{4}x^4+\ldots$$

$$=1-3x^2+4x^4+\ldots$$

$\therefore$ **The Maclaurin series for g(x), where g(x) = cos^6x, up to and including the term in x^4** is

$\cos^6 x \approx 1 - 3x^2 + 4x^4$ **(ans)** [3]

☺ **CheckBack**

If the answer is $\cos^6 x \approx 1 - 3x^2 + 4x^4$, ❶

$\therefore$ Let $x = 0.1$ radian (choose a small value),

LHS of ❶ = $\cos^6\left(0.1^r\right) = 0.9703968827$

RHS of ❶ = $1 - 3(0.1)^2 + 4(0.1)^4 + \ldots$

$= 0.9704$

$\approx$ LHS of ❶

(checked)

(ii)
(a) $\therefore$ Using the answer to **part (i)** to give an approximation for $\int_0^a g(x)\cdot dx$ in terms of a,

$$\int_0^a g(x)\cdot dx$$

$$= \int_0^a \left(1 - 3x^2 + 4x^4\right)\cdot dx = \left[x - x^3 + \tfrac{4}{5}x^4\right]_0^a$$

$$= \left[a - a^3 + \tfrac{4}{5}a^5\right]$$

$\therefore$ **Using the answer to part (i) to give an approximation for $\int_0^a g(x)\cdot dx$ in terms of a** is

$$\int_0^a g(x)\cdot dx = a\left(1 - a^2 + \tfrac{4}{5}a^4\right)$$

$\therefore$ **This approximation in the case where $a = ¼\pi$** is evaluated as

$$\int_0^{\frac{1}{4}\pi} g(x)\cdot dx = \tfrac{1}{4}\pi\left[1 - \left(\tfrac{1}{4}\pi\right)^2 + \tfrac{4}{5}\left(\tfrac{1}{4}\pi\right)^4\right]$$

$= 0.5400 = 0.540$ (3sf) **(ans)** [3]

☺ **CheckBack**

There is no easy CheckBack option for this question.

(checked)

- **Exam Report**

  Quite a large number of candidates did not evaluate the integral expression correctly, such as arriving at $\left[a - a^3 + a^4\right]$. Quite a fair bit of credit was not awarded.

(b) $\therefore$ **Using the calculator to find an accurate value for $\int_0^a g(x)\cdot dx$,**

The value = 0.475 (3sf) **(ans)**

$\therefore$ **The reason why the approximation in part (ii) (a) is not very good** is

The *Maclaurin Series* has an infinite number of terms to evaluate an accurate value for the expression. If the number of terms used is too small, the accuracy of the evaluation (the approximation) would not be very good. **(ans)** [2]

☺ **CheckBack**

There is no easy CheckBack option for this question. Students may try to redo the operation on the calculator.

(checked)

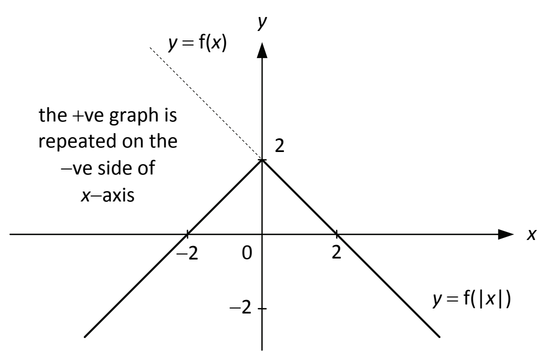
5. [Functions & graphs]

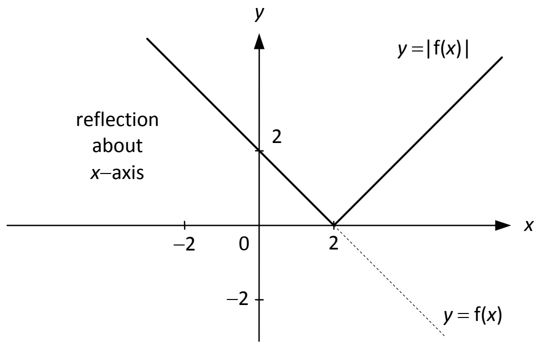
Solution

Given: $f(x) = 2 - x$

(i) **On separate diagrams,**

$\therefore$ **The graphs of $y = f(|x|)$ and $y = |f(x)|$, giving the coordinates of any points where the graphs meet the x– and y– axes** are sketched. **The graphs** are **clearly labelled.**





(ans) [3]

(ii) $\therefore$ **The set of values of *x* for which $f(|x|)=|f(x)|$** is

$\{x:\ 0\le x\le 2,\ \ x\in\mathbb{R}\}$ **(ans)** [1]

- **Exam Report**

  Some candidates suggested the plain answer as $0\le x\le 2$. The answer mark was credited. If the credit had been higher, the answer mark would not have been awarded.

(iii) $\therefore$ To find the exact value of the constant *a* for which $\int_{-1}^{1} f(|x|)\cdot dx=\int_{1}^{a}|f(x)|\cdot dx$:

$$\int_{-1}^{1} f(|x|)\cdot dx = 2\times\frac{2+1}{2}\cdot 1$$

$= 3\ \text{unit}^2$ (trapezium) ❶

$$\int_{1}^{a}|f(x)|\cdot dx = ½+\int_{2}^{a}|f(x)|\cdot dx$$

$$= ½+\int_{2}^{a}(x-2)\cdot dx \quad = ½+\left[\frac{x^2}{2}-2x\right]_2^a$$

$$= ½+\left[\frac{a^2}{2}-2a\right]-\left[\frac{2^2}{2}-2(2)\right]$$

$= ½+\frac{1}{2}a^2-2a-2$

$= ½+\frac{1}{2}a^2-2a-2$ ❷

Equating ❶ and ❷:

$3 = ½+½a^2-2a-2 \ \Rightarrow\ 1½ = ½a^2-2a$

$\Rightarrow\ 1=a^2-4a \ \Rightarrow\ 1+4=a^2-4a+4$

$\Rightarrow\ 5=(a-2)^2 \ \Rightarrow\ \pm\sqrt{5}=(a-2)$

$\Rightarrow\ a=2\pm\sqrt{5}$

The solution of $a=2-\sqrt{5}$ is rejected.

$\therefore$ **The exact value of the constant *a* for which** $\int_{-1}^{1} f(|x|)\cdot dx=\int_{1}^{a}|f(x)|\cdot dx$ is

$a=2+\sqrt{5}$ **(ans)** [3]

☺ **CheckBack**

If the answer is $a=2+\sqrt{5}$,

$\therefore$ Area under the $\int_{1}^{a}|f(x)|\cdot dx$

$= ½+\frac{1}{2}\left(2+\sqrt{5}-2\right)\left(2+\sqrt{5}-2\right)$ (area of Δ)

$= ½+\frac{5}{2} = 3\ \text{units}^2 \equiv \int_{-1}^{1} f(|x|)\cdot dx$

(checked)

- **Exam Report**

  This was considered a simple algebraic manipulation. But, still the number of candidates who gave wrong answers was unusually large. Candidates were advised to be extremely careful about algebraic manipulations about the equality sign.

6. [Sequences & series]

Solution

(i) $\therefore$ **Using the formulae for** $\sin(A\pm B)$,

$\sin(A\pm B)\equiv \sin A\cos B\pm\cos A\sin B$

$\therefore$ **To prove**

$\sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta=2\cos r\theta\sin\frac{1}{2}\theta$ — ❶

LHS of ❶ $= \sin\left(r+\frac{1}{2}\right)\theta-\sin\left(r-\frac{1}{2}\right)\theta$

$= \left(\sin r\theta\cos\frac{1}{2}\theta+\cos r\theta\sin\frac{1}{2}\theta\right) - \left(\sin r\theta\cos\frac{1}{2}\theta-\cos r\theta\sin\frac{1}{2}\theta\right)$

$= 2\cos r\theta\sin\frac{1}{2}\theta$

$=$ RHS of ❶ **(QED)** [2]

☺ **Common misconception**

Q.E.D. does not mean "*quite easily done*", but is an acronym of the Latin phrase *quod erat demonstrandum*, which means "that which was to be demonstrated". The phrase is traditionally placed in its abbreviated form at the end of a mathematical proof or philosophical argument when that which was specified in the enunciation, and in the setting–out, has been exactly restated as the conclusion of the demonstration. The abbreviation thus signals the completion of the proof.

(ii) **Hence,**

In terms of $\sin\left(n+\frac{1}{2}\right)\theta$ **and** $\sin\frac{1}{2}\theta$ **,**

$\therefore$ From **part (i)**,

$$\sin\left(r+\tfrac{1}{2}\right)\theta-\sin\left(r-\tfrac{1}{2}\right)\theta=2\cos r\theta\sin\tfrac{1}{2}\theta$$

$$2\cos r\theta\sin\tfrac{1}{2}\theta=\sin\left(r+\tfrac{1}{2}\right)\theta-\sin\left(r-\tfrac{1}{2}\right)\theta$$

$$\sum_{r=1}^{n}\left(2\cos r\theta\sin\tfrac{1}{2}\theta\right)=\sum_{r=1}^{n}\left(\sin\left(r+\tfrac{1}{2}\right)\theta-\sin\left(r-\tfrac{1}{2}\right)\theta\right)$$

$$\sum_{r=1}^{n}\left(2\cos r\theta\sin\tfrac{1}{2}\theta\right)=\sum_{r=1}^{n}\left(\sin\left(r+\tfrac{1}{2}\right)\theta-\sin\left(r-\tfrac{1}{2}\right)\theta\right)$$

$$=\left[\sin\left(1+\tfrac{1}{2}\right)\theta-\sin\left(1-\tfrac{1}{2}\right)\theta\right]$$
$$+\left[\sin\left(2+\tfrac{1}{2}\right)\theta-\sin\left(2-\tfrac{1}{2}\right)\theta\right]$$
$$+\left[\sin\left(3+\tfrac{1}{2}\right)\theta-\sin\left(3-\tfrac{1}{2}\right)\theta\right]+\ldots$$
$$+\left[\sin\left(n-1+\tfrac{1}{2}\right)\theta-\sin\left(n-1-\tfrac{1}{2}\right)\theta\right]$$
$$+\left[\sin\left(n+\tfrac{1}{2}\right)\theta-\sin\left(n-\tfrac{1}{2}\right)\theta\right]$$
$$=\left[\sin\left(n+\tfrac{1}{2}\right)\theta-\sin\left(\tfrac{1}{2}\right)\theta\right]$$

$$\sum_{r=1}^{n}\left(2\cos r\theta\sin\tfrac{1}{2}\theta\right)=\left[\sin\left(n+\tfrac{1}{2}\right)\theta-\sin\left(\tfrac{1}{2}\right)\theta\right]$$

— ❷

From ❷,

$\therefore$ **A formula for** $\sum_{r=1}^{n}\cos r\theta$ **in terms of** $\sin\left(n+\frac{1}{2}\right)\theta$ **and** $\sin\frac{1}{2}\theta$ is

$$\sum_{r=1}^{n}\cos r\theta=\frac{1}{2\sin\left(\frac{1}{2}\right)\theta}\cdot\left[\sin\left(n+\tfrac{1}{2}\right)\theta-\sin\left(\tfrac{1}{2}\right)\theta\right]$$

$$=\tfrac{1}{2}\left[\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{1}{2}\right)\theta}-1\right]$$ **(ans)** [3]

☺ **CheckBack**

There is no easy CheckBack option for this question.

(checked)

(iii) The summation of a series is defined as

$$\sum_{r=1}^{n}\sin r\theta=\frac{\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

for all positive integers *n*.

$\therefore$ **Proof by the method of mathematical induction:**

Let P_n be

$$\sum_{r=1}^{n}\sin r\theta=\frac{\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}\quad\text{where } n\in\mathbb{Z}^{+}$$

When $n=1$,

LHS of $P_1=\sum_{r=1}^{1}\sin r\theta=\sin(1)\theta=\sin\theta$

RHS of P_1

$$=\frac{\cos\frac{1}{2}\theta-\cos\left(1+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}=\frac{-2\sin\theta\sin\left(-\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

$$=\frac{2\sin\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}=\sin\theta=\text{LHS of } P_1$$

$\therefore P_1$ is true.

Assume P_k is true for some $k\in\mathbb{Z}^{+}$, *i.e.*,

$$\sum_{r=1}^{k}\sin r\theta=\frac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}\quad\text{where}$$

$k\le n\in\mathbb{Z}^{+}$.

When $n=k+1$,

RHS of $P_{k+1}=\dfrac{\cos\frac{1}{2}\theta-\cos\left(k+1+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$

LHS of $P_{k+1}=\sum_{r=1}^{k+1}\sin r\theta$

$$=\sin(k+1)\theta+\sum_{r=1}^{k}\sin r\theta$$

$$=\sin(k+1)\theta+\frac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

$$=\frac{2\sin\frac{1}{2}\theta\sin(k+1)\theta}{2\sin\frac{1}{2}\theta}+\frac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{-2\sin\frac{1}{2}\theta\sin(k+1)\theta}{-2\sin\frac{1}{2}\theta} + \frac{\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\cos(k+1\frac{1}{2})\theta - \cos(k+\frac{1}{2})\theta}{-2\sin\frac{1}{2}\theta} + \frac{\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\cos\frac{1}{2}\theta - \cos(k+1\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\cos\frac{1}{2}\theta - \cos(k+1+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

= RHS of P_{k+1}

$\therefore$ P_{k+1} is true when P_k is true.

$\therefore$ By **mathematical induction**, P_n is true for all $n \in \mathbb{Z}^+$. **(QED)** [6]

☺ **Mathematic induction**

Mathematical induction refers to the method which is commonly used to prove mathematical statements, which refer to a variable natural number.

Steps for mathematical induction:

❶ Let P_n be the statement about a variable natural number *n*.

❷ Prove that the statement is true for *m* where *m* is the least value that *n* can assume, usually $m = 1$, *i.e.*, show that P_m is true.

❸ Assume that the statement is true for $n = k$ and show that the statement is also true for $n = k + 1$, *i.e.*, show that P_k is true $\Rightarrow P_{k+1}$ is true.

❹ Combine ❷ and ❸, $P_m \Rightarrow P_{m+1} \Rightarrow P_{m+2} \Rightarrow \ldots$, and conclude that "By mathematical induction, the statement is true for all $n \in Z^+, n \geq m$."

• **Exam Report**

A candidate suggested the conclusion to proof of the mathematical induction as "Since P_1 is true and P_k is true implies P_{k+1} is true, P_n is true by mathematical induction" is clearly illogical.

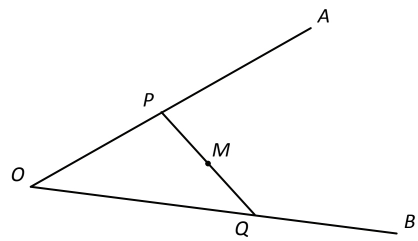
If P_k was already true and $k \in \mathbb{Z}^+$, then there would be no further need to prove anymore.

The proof must clearly state that when P_k is true, P_{k+1} is proved to be true. Since P_1 is proved to be true, it implies P_2 is true. Since P_2 is true, it implies P_3 is also true, and so on. The above statement given by the candidate is clearly on the wrong logical presentation. Another essential statement in the final description must be that "P_n is true for all $n \in \mathbb{Z}^+$".

Prove by mathematical induction is a very strict proof, several marks or the final answer mark were not credited.

7. [Vectors]

Solution



The diagram above shows an origin *O*. The points *A* and *B* are points in space, relative to the origin, such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. A point *P* is on *OA* such that *OP* : *PA* = 1 : 2, while another point *Q* is on *OB* such that *OQ* : *QB* = 3 : 2. As for line segment *PQ*, its mid–point is *M*.

(i) $\therefore$ $\overrightarrow{OM}$**, in terms of a and b,**

$$= \overrightarrow{OQ} + \overrightarrow{QM}$$

$$= \tfrac{3}{5}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{QP} = \tfrac{3}{5}\mathbf{b} + \tfrac{1}{2}\left(\overrightarrow{OP} - \overrightarrow{OQ}\right)$$

$$= \tfrac{3}{5}\mathbf{b} + \tfrac{1}{2}\left(\tfrac{1}{3}\overrightarrow{OA} - \tfrac{3}{5}\mathbf{b}\right) = \tfrac{3}{5}\mathbf{b} + \tfrac{1}{2}\left(\tfrac{1}{3}\mathbf{a} - \tfrac{3}{5}\mathbf{b}\right)$$

$$= \tfrac{3}{5}\mathbf{b} + \tfrac{1}{6}\mathbf{a} - \tfrac{3}{10}\mathbf{b}$$

$$= \tfrac{1}{6}\mathbf{a} + \tfrac{3}{10}\mathbf{b} \quad \textbf{(ans)}$$

$\therefore$ **To show that the area of triangle *OMP* can be written as** $k|\mathbf{a} \times \mathbf{b}|$, where *k* is a constant to be found:

Area of triangle $OMP = \frac{1}{2} \times ab \sin C$

$$= \tfrac{1}{2}\left|\overrightarrow{OP} \times \overrightarrow{OM}\right| = \tfrac{1}{2}\left|\tfrac{1}{3}\mathbf{a} \times \left(\tfrac{1}{6}\mathbf{a} + \tfrac{3}{10}\mathbf{b}\right)\right|$$

$$= \tfrac{1}{2}\left|\tfrac{1}{3}\mathbf{a} \times \tfrac{1}{6}\mathbf{a} + \tfrac{1}{3}\mathbf{a} \times \tfrac{3}{10}\mathbf{b}\right| = \tfrac{1}{2}\left|\mathbf{0} + \tfrac{1}{10}\mathbf{a} \times \mathbf{b}\right|$$

$$= \tfrac{1}{20}|\mathbf{a} \times \mathbf{b}| \quad \textbf{(QED)}$$

where *k* is a constant $= \frac{1}{20}$ **(ans)** [6]

☺ **CheckBack**

There is no easy CheckBack option for this question.

(checked)

(ii) Given further that the vectors **a** and **b** are defined by

$$\mathbf{a} = 2p\mathbf{i} - 6p\mathbf{j} + 3p\mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

where p is a positive constant. The vector **a** is a unit vector.

(a) ∴ To find the exact value of p:

The magnitude of **a**

$$= 1 = \sqrt{(2p)^2 + (-6p)^2 + (3p)^2}$$

$$\Rightarrow \quad 1^2 = (2p)^2 + (-6p)^2 + (3p)^2$$

$$\Rightarrow \quad 1 = 4p^2 + 36p^2 + 9p^2 = 49p^2$$

$$\Rightarrow \quad p = \tfrac{1}{7}$$

∴ **The exact value of *p*** is

$\frac{1}{7}$ unit. **(ans)** [2]

☺ **CheckBack**

If the answer is $\frac{1}{7}$,

$$\therefore \ \mathbf{a} = \tfrac{2}{7}\mathbf{i} - \tfrac{6}{7}\mathbf{j} + \tfrac{3}{7}\mathbf{k}$$

The magnitude of **a** $= \sqrt{\left(\frac{2}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{3}{7}\right)^2}$

$$= \sqrt{\left(\tfrac{4}{49}\right) + \left(\tfrac{36}{49}\right) + \left(\tfrac{9}{49}\right)} = \sqrt{\left(\tfrac{49}{49}\right)}$$

= 1 unit (unit vector) **(checked)**

(b) ∴ **A geometrical interpretation of** $|\mathbf{a} \cdot \mathbf{b}|$ is

The absolute length of the scalar projection of vector **a** onto vector **b** or in another words, it is the absolute length of shadow of vector **a** when it is projected onto vector **b** in **b**'s direction. **(ans)** [1]

• **Exam Report**

Quite a large number of candidates gave the answer as a projection of vector **b** onto vector **a**. Although numerically it is not different from a projection of vector **a** onto vector **b**, the geometrical interpretation however was definitive. This was marked incorrect and the answer mark was not credited.

(b) ∴ $|\mathbf{a} \times \mathbf{b}|$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{7} & -\frac{6}{7} & \frac{3}{7} \\ 1 & 1 & -2 \end{vmatrix} = \begin{array}{l} \left(\left(-\frac{6}{7}\right)\times(-2) - 1\times\frac{3}{7}\right)\mathbf{i} \\ +\left(\frac{3}{7}\times 1 - \frac{2}{7}\times(-2)\right)\mathbf{j} \\ +\left(\frac{2}{7}\times 1 - 1\times\left(-\frac{6}{7}\right)\right)\mathbf{k} \end{array}$$

$$= \tfrac{9}{7}\mathbf{i} + \tfrac{7}{7}\mathbf{j} + \tfrac{8}{7}\mathbf{k}$$

$= \frac{1}{7}(9\mathbf{i} + 7\mathbf{j} + 8\mathbf{k})$ **(ans)** [2]

• **Exam Report**

Quite a number of candidates gave the answer as $\frac{1}{7}\begin{pmatrix} 9 \\ 7 \\ 8 \end{pmatrix}$. This was not credited with the answer mark. The question did not ask for a column vector, as it was never defined in the question. Hence, it cannot be used as part of the answer.

8. [Calculus]

Solution

(i) ∴ $\int \frac{1}{100 - v^2} \cdot dv$ (from formula list)

$$= \tfrac{1}{20}\ln\left|\frac{10 + v}{10 - v}\right| + c \quad (|v| < 10)$$

where c is an integral constant **(ans)** [2]

☺ **CheckBack**

If the answer is $\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right|+c$,

$$\therefore\ \frac{d}{dv}\left[\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right|+c\right]$$

$$=\left[\frac{1}{20}\left(\frac{10-v}{10+v}\right)\cdot\frac{(10-v)(1)-(10+v)(-1)}{(10-v)^2}\right]$$

$$=\left[\frac{1}{20}\left(\frac{10-v}{10+v}\right)\cdot\frac{10-v+10+v}{(10-v)^2}\right]$$

$$=\left[\frac{1}{20}\left(\frac{1}{10+v}\right)\cdot\frac{20}{(10-v)^1}\right]$$

$$=\frac{1}{100-v^2}\quad\textbf{(checked)}$$

(iii) A gravity experiment is made on an air balloon at a height by dropping a stone from that stationary air balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{dv}{dt}=10-0.1v^2 \qquad ❶$$

(a) $\therefore$ To find t in terms of v:

From ❶,

$$\int\frac{dv}{10-0.1v^2}=\int dt\quad\text{(variable separable)}$$

$$\Rightarrow\ \int\frac{10\cdot dv}{100-v^2}=\int 1\cdot dt\quad\text{(From \textbf{part (i)})}$$

$$\Rightarrow\ 10\left(\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right|\right)+c=t\quad(|v|<10)$$

$$\Rightarrow\ t=\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|+c$$

When $t=0$ s, $v=0$ m/s,

$$0=\frac{1}{2}\ln\left|\frac{10+0}{10-0}\right|+c\ \Rightarrow\ c=0$$

$\therefore$ **The time t, in terms of v,** is

$$t=\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|\quad(|v|<10)\quad\textbf{(ans)}$$

☺ **CheckBack**

If the answer is $t=\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$,

$$\therefore\ \frac{d}{dt}[t]=\frac{d}{dt}\left[\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|\right]$$

$$\Rightarrow\ 1=\frac{1}{2}\left(\frac{10-v}{10+v}\right)\left[\frac{(10+v)(\dot v)-(10-v)(-\dot v)}{(10-v)^2}\right]$$

$$\Rightarrow\ 1=\frac{1}{2}\left(\frac{10-v}{10+v}\right)\left[\frac{20}{(10-v)^2}\right]\cdot\dot v$$

$$\Rightarrow\ 1=\left(\frac{10}{100-v^2}\right)\cdot\dot v\ \Rightarrow\ \dot v=\frac{100-v^2}{10}$$

$$\Rightarrow\ \dot v=10-0.1v^2\quad\textbf{(checked)}$$

- **Exam Report**

  Quite a number of candidates gave the erroneous answer as $t=\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right|$. This was not credited with the answer mark. Others did not specify that $|v|<10$, again the answer mark was not credited.

$\therefore$ **Hence, the exact time the stone takes to reach a speed of 5 metres per second** is

$$t_{5m/s}=\frac{1}{2}\ln\left|\frac{10+5}{10-5}\right|$$

$$=\frac{1}{2}\ln\left|\frac{15}{5}\right|$$

$$=\frac{1}{2}\ln 3\ \text{s}\quad\textbf{(ans)}\quad[5]$$

☺ **CheckBack**

If the answer is $t=\frac{1}{2}\ln 3$,

$$\therefore\ \frac{1}{2}\ln 3=\frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$$

$$\Rightarrow\ 3=\frac{10+v}{10-v}\ \Rightarrow\ 30-3v=10+v$$

$$\Rightarrow\ 30-10=4v\ \Rightarrow\ 4v=20$$

$$\Rightarrow\ v=5\ \text{m/s}\quad\textbf{(checked)}$$

(b) $\therefore$ **The speed of the stone after 1 second,** is

$$1\text{ s} = \tfrac{1}{2}\ln\left|\frac{10+v_{1s}}{10-v_{1s}}\right|$$

$$\Rightarrow \quad 2 = \ln\left|\frac{10+v_{1s}}{10-v_{1s}}\right| \quad \Rightarrow \quad \frac{10+v_{1s}}{10-v_{1s}} = e^2$$

$$\Rightarrow \quad 10+v_{1s} = e^2\left(10-v_{1s}\right)$$

$$\Rightarrow \quad v_{1s} + v_{1s}\cdot e^2 = 10\cdot e^2 - 10$$

$$\Rightarrow \quad v_{1s} = 10\cdot\frac{\left(e^2-1\right)}{\left(e^2+1\right)} \quad \textbf{(ans)} \quad [3]$$

☺ **CheckBack**

If the answer is $v_{1s} = 10\cdot\frac{\left(e^2-1\right)}{\left(e^2+1\right)}$,

$$\therefore \ t = \tfrac{1}{2}\ln\left|\frac{10+10\cdot\frac{\left(e^2-1\right)}{\left(e^2+1\right)}}{10-10\cdot\frac{\left(e^2-1\right)}{\left(e^2+1\right)}}\right|$$

$$= \tfrac{1}{2}\ln\left|\frac{1+\frac{\left(e^2-1\right)}{\left(e^2+1\right)}}{1-\frac{\left(e^2-1\right)}{\left(e^2+1\right)}}\right| = \tfrac{1}{2}\ln\left|\frac{\left(e^2+1\right)+\left(e^2-1\right)}{\left(e^2+1\right)-\left(e^2-1\right)}\right|$$

$$= \tfrac{1}{2}\ln\left|\frac{2e^2}{2}\right| = \tfrac{1}{2}\ln\left|e^2\right| = \tfrac{1}{2}\cdot 2\cdot\ln\left|e\right|$$

$$= 1\text{ s} \quad \textbf{(checked)}$$

• **Exam Report**

Quite a number of candidates gave the final answer as 7.62 m/s. This was not credited with the answer mark. The answer expected was an exact expression.

(c) $\therefore$ **For large values of time, the value of the speed of the stone** becomes

$$t = \tfrac{1}{2}\ln\left|\frac{10+v}{10-v}\right| \ \to\ \infty \ \Rightarrow\ 10-v \ \to\ 0^+$$

i.e., almost 10 m/s **(ans)** [2]

9. [Sequences & Series]

Solution

(i) An oil–prospecting company is drilling for oil. Using machine *A*, the depth drilled on the first day is 256 metres. On each subsequent day, the depth drilled is 7 metres less than on the previous day. Drilling continues daily up to including the day when the depth of less than 10 metres is drilled.

$\therefore$ Deduce that,

$T_1 = 256$, $T_2 = 256 - 7\times1$, $T_3 = 256 - 7\times2$, …

$T_n = 256 - 7(n-1)$

First term, $a = 256$; difference, $d = -7$

$\therefore$ **The depth drilled on the 10th day,** is

$T_{10} = 256 - 7(10-1)$

$= 256 - 7\times9 = 193$ m **(ans)**

$\therefore$ Deduce further that,

$T_{n,\,\text{last}} = 256 - 7(n-1) < 10$

$\Rightarrow$ $256 + 7 < 10 + 7n$ $\Rightarrow$ $263 - 10 < 7n$

$\Rightarrow$ $253 < 7n$ $\Rightarrow$ $36.14 < n$

$\Rightarrow$ $n = 37$ (first term over)

$\therefore$ **The total depth when the drilling is completed** is

$$S_n = \frac{n}{2}[a+\ell]$$

$$S_{37} = \tfrac{37}{2}\left[256+256-7\left(37-1\right)\right]$$

$= 4810$ m **(ans)** [6]

(ii) Using machine B, the depth drilled on the first day is also 256 metres. On each subsequent day, the depth drilled is 8/9 of the depth drilled on the previous day.

$\therefore$ Deduce that,

$T_1 = 256$, $T_2 = 256\times\left(\tfrac{8}{9}\right)$, $T_3 = 256\times\left(\tfrac{8}{9}\right)^2$, …

$T_n = 256\times\left(\tfrac{8}{9}\right)^{n-1}$

First term, $a = 256$; ratio, $r = \left(\tfrac{8}{9}\right)$

$\therefore$ **The number of days it takes for the depth drilled to exceed 99% of the theoretical maximum total depth** is

$S_n > 99\% \cdot S_\infty$

$\Rightarrow \dfrac{a(1-r^n)}{1-r} > 0.99 \times \dfrac{a}{1-r}$

$\Rightarrow \dfrac{256\left(1-\left(\frac{8}{9}\right)^n\right)}{1-\left(\frac{8}{9}\right)} > 0.99 \times \dfrac{256}{1-\left(\frac{8}{9}\right)}$

$\Rightarrow 1-\left(\frac{8}{9}\right)^n > 0.99 \quad \Rightarrow \quad \left(\frac{8}{9}\right)^n < 0.01$

$\Rightarrow n \cdot \left|\lg\left(\frac{8}{9}\right)\right| > \left|\lg 0.01\right| \quad \Rightarrow \quad n > \dfrac{\left|\lg 0.01\right|}{\left|\lg\left(\frac{8}{9}\right)\right|}$

$\Rightarrow n > 39.1$

i.e., 40 days **(ans)** [4]

☺ **CheckBack**

There is no easy CheckBack option for this question. Students are advised in this case to substitute the answer as a check. **(checked)**

10. [Complex numbers]

Solution

[Do not use a graphic calculator in answering this question.]

(i) [Showing your working.]

z_1 and z_2 are the roots of the equation $z^2 = -8\text{i}$.

$\therefore z^2 = -8\text{i} = 8e^{\text{i}(2n\pi + \frac{3}{2}\pi)}$

$\Rightarrow z = \sqrt{8}\, e^{\text{i}\frac{(2n\pi + \frac{3}{2}\pi)}{2}}$

$\Rightarrow z = 2\sqrt{2}e^{\text{i}(n\pi + \frac{3}{4}\pi)}$

$\therefore$ **In cartesian form $x + \text{i}y$,**

$z_1 = 2\sqrt{2}e^{\text{i}(\frac{3}{4}\pi)}$

$= 2\sqrt{2}\left[\cos\left(\frac{3}{4}\pi\right) + \text{i}\sin\left(\frac{3}{4}\pi\right)\right]$

$= 2\sqrt{2}\left[-\dfrac{\sqrt{2}}{2} + \text{i}\left(\dfrac{\sqrt{2}}{2}\right)\right]$

$= -2 + 2\text{i}$ **(ans)**

$z_2 = 2\sqrt{2}e^{\text{i}(\pi + \frac{3}{4}\pi)}$

$= 2\sqrt{2}\left[\cos\left(\frac{7}{4}\pi\right) + \text{i}\sin\left(\frac{7}{4}\pi\right)\right]$

$= 2\sqrt{2}\left[\dfrac{\sqrt{2}}{2} + \text{i}\left(-\dfrac{\sqrt{2}}{2}\right)\right]$

$= 2 - 2\text{i}$ **(ans)** [4]

☺ **CheckBack**

If the answer is $z_1 = -2 + 2\text{i}$,

$\therefore z_1^2 = (-2+2\text{i})^2$

$= (-2+2\text{i})(-2+2\text{i})$

$= 4 - 4\text{i} - 4\text{i} - 4$

$= -8\text{i}$

$\therefore z_2^2 = (2-2\text{i})^2$

$= (2-2\text{i})(2-2\text{i})$

$= 4 - 4\text{i} - 4\text{i} - 4$

$= -8\text{i}$ **(checked)**

- **Exam Report**

  Some candidates suggested the conjugate of the root, $-2+2\text{i}$, as the other root, *i.e.*, $-2-2\text{i}$.
  This was a conceptual error and half of the credit was not awarded.

(ii) **Hence, or otherwise,**

Given that w_1 and w_2 are the roots of the equation

$w^2 + 4w + (4+2\text{i}) = 0$ ❶

$\therefore$ From ❶,

$w^2 + 4w + (4+2\text{i}) = 0$

$\Rightarrow w^2 + 4w + 4 = -2\text{i} \quad \Rightarrow \quad (w+2)^2 = -2\text{i}$

$\Rightarrow (w+2)^2 = 2e^{\text{i}(2n\pi + \frac{3}{2}\pi)}$

$\Rightarrow (w+2) = \sqrt{2}e^{\text{i}\frac{(2n\pi + \frac{3}{2}\pi)}{2}}$

$\Rightarrow w = \sqrt{2}e^{\text{i}(n\pi + \frac{3}{4}\pi)} - 2$

$\therefore$ **In cartesian form $x+iy$,**

$$w_1 = \sqrt{2}e^{i\left(\frac{3}{4}\pi\right)} - 2$$
$$= \sqrt{2}\left[\cos\left(\tfrac{3}{4}\pi\right) + i\sin\left(\tfrac{3}{4}\pi\right)\right] - 2$$
$$= \sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] - 2$$
$$= -3 + i \quad \textbf{(ans)}$$

$$w_2 = \sqrt{2}e^{i\left(\pi+\frac{3}{4}\pi\right)} - 2$$
$$= \sqrt{2}\left[\cos\left(\tfrac{7}{4}\pi\right) + i\sin\left(\tfrac{7}{4}\pi\right)\right] - 2$$
$$= \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] - 2$$
$$= -1 - i \quad \textbf{(ans)} \quad [3]$$

☺ **CheckBack**

If the answer is $w_1 = -3 + i$,

$\therefore$ LHS of ❶ $= w_1^2 + 4w_1 + (4+2i)$

$$= (-3+i)^2 + 4(-3+i) + (4+2i)$$
$$= 9 - 6i - 1 - 12 + 4i + 4 + 2i$$
$$= 0 = \text{RHS of ❶}$$

If the answer is $w_2 = -1 - i$,

$\therefore$ LHS of ❶ $= w_2^2 + 4w_2 + (4+2i)$

$$= (-1-i)^2 + 4(-1-i) + (4+2i)$$
$$= 1 + 2i - 1 - 4 - 4i + 4 + 2i$$
$$= 0 = \text{RHS of ❶}$$

(checked)

• **Exam Report**

Some candidates suggested the conjugate of the root, $-3+i$, as the other root, *i.e.*, $-3-i$. This was a conceptual error and the answer mark of the other root was not awarded.

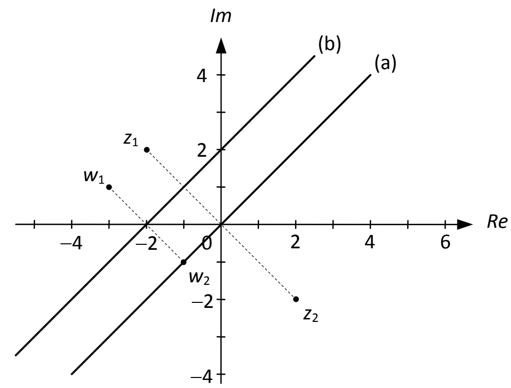
(iii) $\therefore$ **Using a single Argand diagram, the following loci** are sketched.

(a) $\therefore |z - z_1| = |z - z_2|$

A perpendicular bisector between points z_1 and z_2.

(b) $\therefore |z - w_1| = |z - w_2|$

A perpendicular bisector between points w_1 and w_2.



(ans) [2]

(iv) $\therefore$ **A reason why there are no points which lie on both of these loci** is

Both the perpendicular bisectors are of same gradient 1 and parallel to each other. One bisector passes through the origin, while the other is offsetted by 2 horizontal units. Hence, there are no common points. **(ans)** [1]

11. [Vectors]

Solution

In a vector space, it is given that a plane p passes through the points with coordinates A (4, –1, –3), B (–2, –5, 2) and C (4, –3, –2).

(i) $\therefore$ To find the cartesian equation of plane, p:

Perpendicular vector **n** to plane, p:

$$BA \times CB = (6, 4, -5) \times (-6, -2, 4)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 4 & -5 \\ -6 & -2 & 4 \end{vmatrix}$$
$$= \begin{array}{l} (4\times4-(-2)\times(-5))\mathbf{i} \\ -(6\times4-(-6)\times(-5))\mathbf{j} \\ +(6\times(-2)-(-6)\times4)\mathbf{k} \end{array}$$

$= 6\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}$

Equation of plane, *p*:

$$r \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = D$$

Substitute point A into the plane, *p*,

$$\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = D$$

$\Rightarrow \quad 4 \times 6 + (-1) \times 6 + (-3) \times 12 = D$

$\Rightarrow \quad -18 = D$

$\therefore$ **The cartesian equation of plane, *p*,**

$$r \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = -18 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = -18$$

$\Rightarrow \quad 6x + 6y + 12z = -18$

$\Rightarrow \quad x + y + 2z = -3$ **(ans)** [4] ❶

A line ℓ_1, with cartesian equation $\frac{x-1}{2} = \frac{y-2}{-4} = \frac{z+3}{1}$, intersects with another line ℓ_2, with cartesian equation $\frac{x+2}{1} = \frac{y-1}{5} = \frac{z-3}{k}$, where *k* is a constant.

(ii) $\therefore$ To find the value of *k*:

$$\ell_1 : \quad r = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \text{ where } \alpha \text{ is a variable}$$

$$\ell_2 : \quad r = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 5 \\ k \end{pmatrix}, \text{ where } \beta \text{ is a variable}$$

For the lines to intersect,

$1 + 2\alpha = -2 + \beta$ ❷

$2 - 4\alpha = 1 + 5\beta \quad \Rightarrow \quad 1 - 2\alpha = ½ + \frac{5}{2}\beta$ ❸

$-3 + \alpha = 3 + k\beta$ ❹

❷+❸:

$2 = -\frac{3}{2} + \frac{7}{2}\beta \quad \Rightarrow \quad \beta = 1, \ \alpha = -1$

$\therefore$ **The value of *k*** is

❹: $-3 - 1 = 3 + k \quad \Rightarrow \quad k = -7$ **(ans)** [4]

☺ **CheckBack**

There is no easy CheckBack option for this question. Students are advised in this case to substitute the answer as a check. **(checked)**

(iii) $\therefore$ **To show that ℓ_1 lies in the plane, *p*:**

Consider ❶,

$$\ell_1 \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = -18$$ ❺

LHS of ❺ $= \ell_1 \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}$

$$= \begin{pmatrix} 1+2\alpha \\ 2-4\alpha \\ -3+\alpha \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}$$

$= 6(1+2\alpha) + 6(2-4\alpha) + 12(-3+\alpha)$

$= (6+12\alpha) + (12-24\alpha) + (-36+12\alpha)$

$= -18$ = RHS of ❺ **(QED)**

$\therefore$ To find the coordinates of the point at which ℓ_2 intersects *p*:

Consider ❶:

$$\ell_2 \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = -18$$

$$\Rightarrow \quad \begin{pmatrix} -2+\beta \\ 1+5\beta \\ 3-7\beta \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = -18$$

$\Rightarrow \quad 6(-2+\beta) + 6(1+5\beta) + 12(3-7\beta) = -18$

$\Rightarrow$
$(-12+6\beta) + (6+30\beta) + (36-84\beta) = -18$

$\Rightarrow \quad 30 - 48\beta = -18 \quad \Rightarrow \quad 48 = 48\beta$

$\Rightarrow \quad \beta = 1$

$\therefore$ **The coordinates of the point at which ℓ_2 intersects *p* are**

$$r = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + (1) \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} = (-1, 6, -4)$$ **(ans)** [4]

☺ **CheckBack**

If the answer is $(-1, 6, -4)$,

$\therefore$ LHS of ❶ $= \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}$

$= -6 + 36 - 48$

$= -18$ = RHS of ❶

(checked)

(iv) $\therefore$ To find the acute angle between ℓ_2 and p:

Consider,

$$\begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix}\right| \cos\theta$$

$$6 + 30 - 84 = \left|\sqrt{1 + 25 + 49}\right| \cdot \left|\sqrt{36 + 36 + 144}\right| \cos\theta$$

$$-48 = \left|\sqrt{75}\right| \cdot \left|\sqrt{216}\right| \cos\theta$$

$$\cos\theta = 112.15°$$

$\therefore$ **The acute angle between ℓ_2 and p** is

$112.15° - 90° = 22.2°$ (3sf) **(ans)** [3]