2.1 Summation of series

Questions

[2013(Pure.II).Q1]

(i) Find the first three terms when $(2+3x)^6$ is expanded in ascending powers of *x*. [3]

(ii) In the expansion of $(1+ax)(2+3x)^6$, the coefficient of x^2 is zero. Find the value of *a*. [2]

ANS: (i) $64+576x+2160x^2$ (ii) $-\frac{2160}{576}$

[Teachers' Comments:] *The fact that the answer was given as part of the question enabled many candidates to obtain full marks as they were able to identify errors, usually sign errors and subsequently correct them. It should be noted that if a candidate fails to obtain a given answer and cannot see where they went wrong, it is far better to leave their work unaltered than to try to contrive to obtain the given answer by incorrect means. There are very often method marks available which candidates are otherwise unable to obtain if they use an incorrect method.*

[2013(Pure.II).Q7]

Let $\mathrm{f}(x)=\dfrac{2x^2-7x-1}{(x-2)(x^2+3)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of *x*, up to and including the term in x^2. [5]

ANS: (i) $\mathrm{f}(x)=-\dfrac{1}{(x-2)}+\dfrac{3x-1}{(x^2+3)}$ (ii) $\frac{1}{6}+\frac{5}{4}x+\frac{17}{72}x^2$

[2012(Pure.II).Q4]

(i) Find the first 3 terms in the expansion of $(2x-x^2)^6$ in ascending powers of x. [3]

(ii) Hence find the coefficient of x^8 in the expansion of $(2+x)(2x-x^2)^6$. [2]

ANS: (i) $64x^6-192x^7+240x^8$ (ii) 288

[2012(Pure.II).Q4]

(i) When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascending powers of x, the coefficients of x and x^3 are equal.

(ii)

(iii) (i) Find the exact value of a. [4]

(iv) (ii) When a has this value, obtain the expansion up to and including the term in x^2, simplifying the coefficients. [3]

ANS: (i) $a=\dfrac{1}{\sqrt{2}}$ (ii) $1-\sqrt{2}x+\frac{3}{2}x^2$

1. (i) Expand $(4+y)^{\frac{1}{2}}$ in ascending powers of *y* up to and including the term y^3, simplifying the coefficients.

(ii) In the expansion of $(4+8x+cx^2)^{\frac{1}{2}}$, where *c* is a constant, the coefficient of x^3 is 2. Substituting $(8x+cx^2)$ as *y*, find the value of *c*.

ANS: (i) $2+\frac{1}{4}y-\frac{1}{64}y^2+\frac{1}{512}y^3$ (ii) $c=-4$

2. (i) Prove, by mathematical induction, that $1^2+3^2+5^2+\dots+(2n-1)^2=\frac{n}{3}(2n-1)(2n+1)$, for $n\in\mathbb{Z}^+$.

(ii) Find an expression, in terms of *n*, for $2^2+4^2+6^2+\dots+(2n)^2$

(iii) Hence or otherwise, show that, when *n* is even, $\sum_{r=1}^{n}(-1)^r r^2=\dfrac{n(n+1)}{2}$

ANS: (ii) $\dfrac{2n}{3}(2n+1)(n+1)$

3. (i) Prove by induction that

$$\sum_{r=1}^{n}(1+r^2)r!=n(n!)(n+1)$$

(ii) Hence, or otherwise, find an expression in simplified form for $\sum_{r=1}^{n}(n!+r!)(1+r^2)$.

ANS: (ii) $n!\left(\dfrac{n}{6}\right)(2n^2+9n+13)$

4. Show that $\sum_{r=n}^{2n}(n+4r)=7n(n+1)$.

Hence find the least value of n such that $\sum_{r=n}^{2n}(n+4r)$ exceeds 700.

ANS: 10

5. Dr Math's son Junior has reached school going age. Dr. Math proposed 2 types of pocket money scheme for his son:

Scheme I: \$20 dollars in Week 1
\$22 dollars in Week 2…and so on, increasing the pocket money by a constant factor of \$2 each week.

Scheme II: \$20 dollars in Week 1
\$21 dollars in Week 2…and so on, increasing the pocket money by a constant factor of $\frac{21}{20}$ each week.

(i) Find separately the total amount in dollars he would receive in n weeks for Schemes I and II, leaving your answers in the simplest form.

(ii) Assuming that a school year consists of 36 "5 day" weeks, which scheme do you think Junior will choose? Justify your answer.

(iii) Answer the following based on your choice of scheme in (ii). Junior Math wants to save up some of his pocket money to buy the latest game console which costs \$400. If he disciplines himself to spend just \$1.50 per school day, which is the earliest week Junior will be able to save enough to buy the game console?

6. Find the binomial expansion of $\sqrt{1+2x}$ up to and including the term in x^3, simplifying the coefficient. State the values of x for which this expansion is valid.

7. (i) Find $\sum_{r=1}^{2n}\ln(r)^2$ in terms of n.

(ii) Prove by induction that

$$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=\frac{n(n+2)}{(n+1)^2}.$$

8. (i) Prove by induction that

$$\sum_{r=1}^{n}\frac{r2^r}{(r+1)(r+2)}=\frac{2^{n+1}}{n+2}-1.$$

(ii) Hence express $\sum_{r=1}^{n}\frac{(r+1)2^r}{(r+2)(r+3)}$ in the form $\frac{2^{n+1}}{n+A}+B$, where A and B are constants to be determined.

9. (i) Express $\sum_{r=1}^{n}(2r-1)^2$ in terms of n.

(ii) The terms of a finite sequence are odd squares not exceeding 5,000:

$1^2, 3^2, 5^2, 7^2, \ldots, k^2$

a. Find the maximum value of k.

b. With this value of k, find the sum of the terms in this finite sequence which are not divisible by 3.

10. (i) Prove by induction that $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1}$ for all positive integers n.

(ii) Hence deduce the sum to infinity of the series

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\ldots.$$

11. (i) Prove by induction that for all positive integers n,

$$\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots+\frac{n+1}{2^n}=3-\left(\frac{1}{2}\right)^n(3+n).$$

(ii) Hence, find $\frac{r+1}{2^r}+\frac{r+2}{2^{r+1}}+\frac{r+3}{2^{r+2}}+\ldots+\frac{2r+1}{2^{2r}}$ in terms of r.

ANS: (ii) $\left(\frac{1}{2}\right)^{r-1}(2+r)-\left(\frac{1}{2}\right)^{2r}(3+2r)$

12. Find an expression for the nth term of the series $2+22+222+2222+\ldots$ and deduce that the sum of the first n terms of the series is $\frac{20}{81}(10^n-1)-\frac{2n}{9}$.

[2012(Pure.II).Q1]

The first term of an arithmetic progression is 61 and the second term is 57. The sum of the first n terms is n. Find the value of the positive integer n. [4]

ANS: 31

1. The geometric progression 1, 2, 4, 8, 16, 32, 64,… is arranged in rows in the following way:

1

2, 4

8, 16, 32

…

The first row consists of one element, the second row consists of the next two elements, the third row consists of the next three elements and so on. In general, the n^{th} row consists of n elements.

Find out

(i) the first term in the n^{th} row;

(ii) the sum of all the terms from the 10^{th} to 20^{th} row.

2. The sequence S, 3, 1, 12, 4, 48, 16, 192, … is formed by using the terms of the following two geometric progressions:

GP_1 : 1, 4, 16, 64, … and

GP_2 : 3, 12, 48, 192, … .

The odd-numbered terms of S are from GP_2 and the even-numbered terms of S are from GP_1 .

The n^{th} term of this new series is 3145728.

Find

(i) the value of n,

(ii) using the value of n in part (i), the sum of the first n terms of the new series.

3. Given that the 97^{th} and the 100^{th} terms of a geometric progression are $3x+x^2$ and $24x^3+8x^4$ respectively, find, in terms of x (where $x \neq 0$), the common ratio of this geometric progression.

ANS: $r = 2x^{\frac{2}{3}}$

4. (i) The first term of an arithmetic progression s 3 and the twenty-fifth term of the series is 15. Find the common difference. Find also the least value of n such that the sum of the first n terms of the series exceeds 2005.

(ii) The sum of infinity of a geometric series is equal to nine times the sum of the first nine terms. Calculate the value of the common ratio, giving your answers to 3 significant figures.

5. (i) The sum of the first 20 terms of an arithmetic series is 45 and the sum of the first 45 terms of the series is $382\frac{1}{2}$. Find the number of terms in the series where the value of each term is less than 20.

(ii) Show that the r^{th} term of the series $3+33+333+3333+\ldots$ can be expressed as $\frac{1}{3}\left(10^r-1\right)$. By using this result, or otherwise, show that the sum of the first n terms, S_n is $\frac{1}{27}\left(10^{n+1}-9n-10\right)$.

6. Kenny laid out $2k-1$ bags in a row and divided 961 marbles among them. There are x marbles in the first bag. For each subsequent bag, the number of marbles is twice the number in the previous bag. Given that there 124 marbles in the kth bag, find the values of k and x.

7. (i) The sum of the first n terms of a series is given by the expression $3-3^{1-n}$. Show that the series is a geometric series.

(ii) Hence find the least value of k such that the sum of the series from the $\left(\frac{k}{2}\right)^{th}$ term onwards is less than $\frac{1}{30}$.

8. The terms of the sequence $a_1, a_2, a_3, \ldots, a_n$ form an arithmetic progression with common difference, d, 2. Given further that $b_r = \left(\frac{1}{3}\right)^{a_r}$, for $r = 1,2,3,\ldots,n$, show that the sequence $b_1, b_2, b_3, \ldots, b_n$ is geometric.

Answer keys:

2.1

1. $2+\frac{y}{4}-\frac{y^2}{64}+\frac{y^3}{512}$; $c=-4$
2. $\frac{2n}{3}(n+1)(2n+1)$
3. $\frac{n}{6}(n!)(2n^2+9n+13)$
4. Least value of n is 10
5. For Scheme I, $S_n=n(n+19)$;
   For Scheme II,
   $S_n=400\left[\left(\frac{21}{20}\right)^n-1\right]$;
   Scheme I; Earliest week is 16th week
6. $1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$ for $-\frac{1}{2}<x<\frac{1}{2}$
7. $2\ln(2n)!$
8. $A=3, B=-\frac{2}{3}$
9. $\frac{1}{3}n(4n^2-1)$; max value of k is 69; 36,455
10. $\frac{1}{2}$
11. $\left(\frac{1}{2}\right)^{r-1}(r+2)-\left(\frac{1}{2}\right)^{2r}(2r+3)$
12. $\frac{2}{9}(10^n-1)$
    $1+2x+\frac{5}{2}x^2+4x^3+\ldots$ for $|x|<\frac{1}{2}$; 3.32
13. $\frac{1}{222}$
14. $A=25$; $B=2$; $C=1$
15. $\frac{1}{4}(5^k-1)+k(k+1)$
19. $1+2x+3x^2+4x^3+\ldots$ for $|x|<1$; 12
20. $\frac{1}{12}$
21. $(n-1)\left(\frac{5}{2}n+8\right)+\frac{1}{100}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]$
22. $\frac{49}{500}$
23. $\ln\left(\frac{2m-1}{4m+1}\right)$
24. $\frac{1}{3}n(n+1)(n+2)$; 1
26. For $|x|>1$, $S_\infty=\frac{1}{x-1}$,
    For $|x|<1$, $S_\infty=\frac{x}{x-1}$
27. $\frac{N}{3}(N+2)(N+4)$
28. $\frac{9(9^k-1)}{8}-k$
30. $2n(2^{n+1})-1(2^n-1)$ or $(2n-1)2^{n+1}+2$
34. $\frac{e^2}{e^3-1}$
37. 1809710
38. $\frac{16949}{7600}$
40. e^{400}
41. $|x|<4$
42. 401, 400
43. 8
44. $|x|>1$, $(-1)^r(r-1)$
45. 0.667
47. $165\frac{1}{3}$
48. $4+8n$; 250
51. $4^{r+1}+(-1)^r\left(\frac{1}{2}\right)^{r+1}$
52. 2094445
53. 3
54. $-4<x<1$
57. 220
58. 4
59. $\frac{381}{512}$
63. 5050
64. 2; $1-\frac{5}{4}x+\frac{5}{4}x^2+\ldots$; $\frac{1}{3}\pi^2-2$
66. 1, 2
67. 76394340
69. $\frac{477}{506}$
71. $\frac{45}{28}$
75. $A=1$, $B=-1$
77. 6
78. $\frac{1536}{623}$
81. 3, 2; $|x|<3$
85. $\frac{1}{2}\left[1-\frac{1}{2n+1}\right]$; $\frac{1}{2}$
86. 3, 7, 15, 31; $u_n=2^{n+1}-1$
87. $2m$
88. $|x|<\frac{1}{2}$; $n^2 2^{n-1}$
89. $\frac{1+\sqrt{29}}{2}$
90. $|x|<4$, 72
91. $\frac{n+2}{2(n+1)}$
92. $6-\frac{n^2+4n+6}{2n}$
95. $\frac{n}{2n+1}$
96. $\frac{n}{n+1}$
97. $1-(N+1)^{-2}$
98. $|x|<\frac{1}{4}$
99. $\frac{1-\sqrt{6}}{5}$
101. $\frac{4n}{4n^2-1}$; $\frac{1}{4}$
102. $|x|<1$
103. 1
104. $\frac{3}{2}+\frac{1}{n+1}-\frac{3}{n+2}-\frac{3}{n+3}$; $\frac{6805}{1771}$
106. $-1<x<1$
107. $\frac{1}{3}$

2.2

1. $T_1 = 2^{\frac{(n-1)n}{2}}$; $2^{45}\left(2^{165}-1\right)$
2. $n = 21$; 4,543,828
3. $r = 2x^{\frac{2}{3}}$
4. Least value of *n* = 85;
   $r \approx 0.987$ to 3 s.f.
5. No. of terms less than 20 = 45
6. $k = 3$; $x = 31$
7. Least value of *k* = 12 (integer)
9. $\dfrac{10^n - 1}{900} + \dfrac{n}{2}\left(3995 + n\right)$
10. $a = 1$; $d = 2$
11. $r = \frac{1}{4}$
12. $a = \frac{1}{10}$; $r = -\frac{1}{4}$
13. $n = 20$ or $n = -\frac{61}{3}$ (N.A.)
14. $0 < a < 2$; Least value of *n* = 22
15. First term = 32;
    $S_n = \frac{128}{3}\left[1 - \left(\frac{1}{4}\right)^n\right]$
16. $T_n = 2n + 1$; $\frac{409}{990}$
17. $a = \sqrt{3}$; $30 - \log_3 2$
18. $k = \frac{4}{3}$; $\frac{1}{6}n\left(n+1\right)\left(4n-1\right)$
19. $r = 0.96$
20. $r = 3$; $\dfrac{2\pi}{63}$ rad
21. $2^{n+1} - 1 + \frac{5}{2}n\left(n+1\right)$
22. $a = 3$; $d = 4$
23. $A = 4$; 500, 501, 502
24. $\frac{3}{2}\left(3^{n-1} - 1\right) - \frac{1}{6}\left(n-2\right)\left(n-1\right)\left(2n-3\right)$
25. $r = -\frac{1}{2}$; $S_\infty = \frac{4}{3}$; $k = \frac{3}{2}\left(-\frac{1}{2}\right)^{n-1}$
26. 3 m^2 ; $T_4 = -\frac{1}{3}$
27. 87
28. 12
29. $-2 < x < 2$, $S_\infty = \dfrac{4}{2-x}$
30. -2.5 , 158865
31. 2, 3
32. -6
33. 174225
35. $-275a$
36. 210
37. $-\frac{1}{3}$
38. 9, 2, 1
39. 1 hour; $5\frac{1}{4}$; 33rd
40. 1 or 4
41. $a = 2, r = \frac{1}{3}$
42. 5
43. $n\ln y$, $\frac{n(n+1)}{2}\ln y$;
    $n\ln x + \frac{n(n+1)}{2}\ln y$
44. 7
45. $-265 + 95\sqrt{5}$
47. $2(3^{n-1}) - 1$, $2(3^{n-1}) - 5$
48. 12; 28.5 m
49. $k = \frac{1+\sqrt{5}}{2}$; $u_n = 2^{n-2}$
50. $\frac{1}{2}$
51. \$3000
52. $\left(\frac{2}{3}, 0\right)$
53. 5, $3\frac{2}{3}$, $4\frac{1}{5}$; $l = 4$
54. $\frac{\sqrt{17}-3}{2}$
55. 3, $\frac{4}{3}$, 1, $\frac{6}{7}$; $a = 2$, $b = -1$
56. $\frac{2}{3}$; 12
57. 41
58. $\frac{4}{3-\sqrt{5}}$
59. -0.586 , -0.820 ; -1
60. $1000 + 2n$; 12; \$4790
61. $\frac{3}{4}$; 3
62. $102 - 3n$
63. 19; $3 + 4n$; 21
64. 0.2; 12
65. -3 , 8; 30050
66. 8 months
67. 2.5
68. \$800
69. 6
70. 400
71. -3951.28 ; 62
72. 90
73. -12 ; -32